

1. (40%) Solve $y(x)$ for the following ordinary differential equation:

(a) $y''+3y'+2y = 0$ (10%)

(b) $y''+4y = 2\sin 2x$ (10%)

(c) $x^2 y'' - 5xy' + 8y = 0$, with $y(1) = 2$ and $y'(1) = 6$ (10%)

(d) $xyy' - y^2 = x^2$, with $y(2) = 0$ (10%) Hint: use $u=y/x$

2. (15%) Derive the Laplace transform for the function $\{\sin kt\}$ is

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$$

3. (20%) The eigenvalues of the matrix $M = \begin{bmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{bmatrix}$ are $-3, 1$ and 5 .

(a) Find the values of a and b (assume: $a > 0$ and $b > 0$) (10%)

(b) Find the normalized eigenvectors for matrix M . (10%)

4. (25%) Using the Fourier series to expand the following function:

(a) $f(x) = x^2$, with $-\pi < x \leq \pi$ (15%) (b) if $x = \pi$, please show that: $\sum_{n=1}^{\infty} \frac{1}{n^2} = ?$ (10%)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ with } -L < x < L$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, 3, \dots \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

$$\int x^2 \sin ax \, dx = \frac{2x}{a^2} \sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a} \right) \cos ax + C$$

$$\int x^2 \cos ax \, dx = \frac{2x}{a^2} \cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3} \right) \sin ax + C$$