行政院國家科學委員會專題研究計畫　成果報告

奧地利單人牌戲中循環的深入研究

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一、 中文摘要

Akin 和 Davis 提出了奧地利單人牌戲。此遊戲是這樣進行的：先將一疊 n 張紙牌分成幾堆，使得每堆的紙牌張數不大於 L，其中 L 是某一個固定的整數。將其中某一特殊堆稱作銀行，保留在旁邊。此遊戲的每次移動包含兩個步驟：首先，從每一普通堆中取出一張牌存入銀行；然後，從銀行裏不斷地分出若干堆，每一堆恰好是 L 張紙牌，直到銀行裏小於 L 張為止（也有可能銀行裏的紙牌剛好全部被取出）。

給定整數 L 和任意一組已經分成若干堆的 n 張紙牌，我們依照上述規重複地移動，最後一定會形成一個循環。例如：從（0;4,3）開始，其中括號裏的第一個數字代表銀行，重複地移動後，依序得到（2;3,2),（0;4,2,1),（3;3,1),（1;4,2),（3;3,1）‧‧‧，形成了(3;3,1),（1;4,2),（3;3,1），（1;4,2)‧‧‧的循環。

對於奧地利單人牌戲，Akin 和 Davis 提出問題：如何描述這遊戲中形成的循環。他們也猜測：給定紙牌張數 n 和整數 L，循環的個數是 1。

在先前關於奧地利單人牌戲的循環的研究中(國科會研究計劃 NSC 91- 2115- M- 034- 001）中，我們證明了，對於一個固定的整數 L，不須要考慮任意的紙牌張數，我們只要考慮 n 張紙牌的奧地利單人牌戲即可‧其中 n 小於或等於 $1+2+ \ldots + L$。 更進一步，我們證明了當 $n + n' = 1+2+ \ldots + L + (L-1)$ 時，n 張紙牌的循環和 n'張紙牌的循環之間存在著一對一的對應關係。

在此計劃中，我們將研究 n 張紙牌的奧地利單人牌戲，其中 n 小於或等於$(1+2+ \ldots + L+(L-1))/2$。我們將試著研究在牌戲過程中產生的銀行數列的規律，並試著描述牌戲中的循環。

關鍵詞： 奧地利單人牌戲，循環

二、 英文摘要

Akin and Davis introduced Austrian Solitaire. It proceeds as follows: Start by laying out piles from a deck of n cards such that each pile has size $\leq L$, where L is some fixed integer. One special pile called the bank is reserved on the side. A move of this game consists of two steps. First, remove one card from each ordinary pile and put it in the bank. Now from the bank lay out new piles of size exactly equal to L, continuing until the size of the bank is $< L$ (including the possibility of exhausting the bank). The operation is repeated over and over.

Starting from any division into piles, one always reaches some cycle of partitions of n. For example, starting from (0;4,3), where the first number in the parentheses corresponds to the bank, the successors are (2;3,2), (0;4,2,1), (3;3,1), (1;4,2), (3;3,1)…. Note that (3;3,1), (1;4,2), (3;3,1) , (1;4,2)‧‧‧form a cycle.

Akin and Davis suggested the problem of describing the cycles in Austrian Solitaire. In

particular, they conjectured that for any fixed deck size n and fixed integer L, there is a unique cycle.

In previous study on cycles in Austrian Solitaire (Project NSC 91-2115-M-034-001), we proved that, for a fixed integer L, instead of considering arbitrary number of cards, it is enough to consider Austrian Solitaire on n cards, where n is at most 1+2+ … + L. Further, we prove that if $n + n' = 1+2+ \ldots + L + (L-1)$, there is a one to one correspondence between the cycles of n cards and the cycles of n' cards.

In this project, we study Austrian Solitaire on n cards, where n is at most $(1+2+ \ldots + L + (L-1))/2$. We investigate the patterns of the sequences of banks, and try to describe the cycles.

Keywords: Austrian Solitaire, cycle

三、 報告內容

(一) 前言及研究目的：

Akin and Davis [1] introduced *Austrian Solitaire*. (The idea for the game arose when they were reading a discussion of the so-called Austrian school of capital theory.) It proceeds as follows: Start by laying out piles from a deck of n cards such that each pile has size $\leqq L$, where L is some fixed integer. One special pile called the *bank* is reserved on the side. A move A_L of this game consists of two steps. First, remove one card from each ordinary pile and put it in the bank. Now from the bank lay out new piles of size exactly equal to L, continuing until the size of the bank is $< L$ (including the possibility of exhausting the bank). The operation A_L is repeated over and over.

Assume n and L are positive integers and $n \geq L$. We say $\lambda = (\lambda_0; \lambda_1, \lambda_2, \ldots, \lambda_s)$ is an L-partition of n (with s parts), provided $0 \leq \lambda_0 < L$, $L \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 1$, and these integers λ_i add up to n. Then each λ corresponds to a stage in Austrian Solitaire with the bank λ_0 and ordinary piles λ_i, $1 \leq i \leq s$. So Austrian Solitaire can be viewed as a way of changing one L-partition of n into another. For instance, if $L = 4$ and $\lambda = (0;4,3)$, then the successors of λ are $A_L(\lambda) = (2;3,2)$, $A_L^{(2)}(\lambda) = (0;4,2,1)$, $A_L^{(3)}(\lambda) = (3;3,1)$, $A_L^{(4)}(\lambda) = (1;4,2)$, $A_L^{(5)}(\lambda) = (3;3,1)$,…. Note that (3;3,1), (1;4,2), (3;3,1), (1;4,2), … form a cycle.

This game has the following economic interpretation: Think of the ordinary piles as machines. Each machine has, when new, a life of exactly L years. The size of a pile is the number of productive years left for a particular machine. Each year it ages one year (and so one card is removed from the pile). For each machine on line the company deposits 1/ L of its cost into the bank as a sinking fund. Then it buys as many new machines as it can afford, and the remaining funds are left in the bank until next year.

For Austrian Solitaire, Akin and Davis[1] proposed

Conjecture 1. *For any fixed deck size n and fixed integer L, there is a unique cycle.*

We say an L-partition of n is *cyclic* if $A_L^{(i)}(\lambda) = \lambda$ for some $i > 0$.

Note that $\lambda = (0; 5, 2)$ is a cyclic 5-partition of 7 and $\lambda' = (0;5,5,4,3,2,2,1)$ is a cyclic 5-partition of 22. In general, we proved in previous project (NSC 91 – 2115 – M – 034 – 001)

Proposition 2. *If* $n' > 1+2+ \ldots + L$, *then every cyclic L-partition of* n' *consists of at least one part of i, for i* $= 1, \ldots, L$.

Thus there is a one to one correspondence between the cyclic L-partitions of n' *and* the cyclic L-partitions of $n' - (1+2+ \ldots + L)$. Further, the number of cycles for n' cards equals the number of cycles for $n' - (1+2+ \ldots + L)$ cards. (This number should be one, if Conjecture 1 is true.)

Note that $\lambda = (0; 5, 2)$ is a cyclic 5-partition of 7 and $\lambda' = (4;4,3,1)$ is a cyclic 5-partition of 12. Let $n + n' = 1+2+ \ldots + L + (L - 1)$. Let $\lambda = (\lambda_0; \lambda_1, \lambda_2, \ldots, \lambda_s)$ be a cyclic L-partition of n with $\lambda_i \neq \lambda_j$ for all $i > j > 0$. We say $\lambda' = (L - 1 - \lambda_0; \lambda'_1, \lambda'_2, \ldots, \lambda'_{L-s})$, is the *complement* of λ if $\{ \lambda'_1, \lambda'_2, \ldots, \lambda'_{L-s} \} = \{ 1, 2, \ldots, L \} \setminus \{ \lambda_1, \lambda_2, \ldots, \lambda_s \}$.

We proved in previous project

Proposition 3. *Let* $n < 1+2+ \ldots + L + (L - 1)$. *If* $\lambda = (\lambda_0; \lambda_1, \lambda_2, \ldots, \lambda_s)$ *is a cyclic L-partition of n, then* $\lambda_i \neq \lambda_j$ *for all i > j > 0.*

Thus, every cyclic L-partition of n has a complement, and hence there is a one to one correspondence between the cyclic L-partitions of n and the cyclic L-partitions of of n' if $n + n' = 1+2+ \ldots + L + (L - 1)$.

The purpose of this project is to study Austrian Solitaire on n cards, where n is at most $(1+2+ \ldots + L + (L - 1))/2$. We investigate the patterns of the sequences of banks, and try to describe the cycles in Austrian Solitaire.

(二) 文獻探討及研究方法

For $L = 7$, Austrian Solitaire on 15 cards has a unique cycle (0; 7, 5, 3), (3; 6, 4, 2), (6; 5, 3, 1), (2; 7, 4, 2), (5; 6, 3, 1), (1; 7, 5, 2), (4; 6, 4, 1), (0; 7, 5, 3),…. We observe the "bank sequence": 0,3,6,2,5,1,4,0, … can be divided into 3 increasing subsequences <0,3,6>, <2,5> and <1,4>, the numbers in these 3 subsequences satisfy $2 - 0 = 5 - 3 = 1 - 6$ (mod

7), and 1 – 2 = 4 – 5 (mod 7). In general, we propose the following conjecture for the bank sequence in Austrian Solitaire

Conjecture 4. Let $b_1, b_2, \ldots, b_n, b_1, \ldots$ be a cycle in the bank sequence.
If $b_n > b_1$, $b_1 < b_2 < \ldots < b_s$, $b_s > b_{s+1}$, $b_m > b_{m+1}$, and $b_{m+1} < b_{m+2}$, then
(1) $b_{m+1} - b_1 = b_{m+2} - b_2 = \ldots = b_{m+s} - b_s \pmod{L}$;
(2) if $b_1 \neq b_{m+1}$ then the intersection of the sets $\{ b_1, b_2, \ldots, b_s \}$ and $\{ b_{m+1}, b_{m+2}, \ldots, b_{m+s-1} \}$ is the empty set.

For a cyclic L-partition λ of n, the *period* of λ is the smallest positive integer k satisfying $A_L^{(k)}(\lambda) = \lambda$. For example, when $L = 5$ the period of $\lambda = (0; 5, 2)$ is 3. We propose

Conjecture 5. *For any cyclic L-partition λ of n, the period of λ is at most L.*

The goal of this project is to prove or disprove the conjectures listed above. We also try to describe the cycles in Austrian Solitaire.

(三) 結果與討論：

Conjecture 4 is true for most cases we investigated. However, we found a counterexample: For $L = 6$ and $n = 20$, we consider the cyclic L-partition $\lambda = (0; 7,6,4,2,1)$, the bank sequence of λ = (0, 5, 2, 6, 3, 0,...) does not satisfy statement (1) in Conjecture 4.

Though Conjecture 4 is disproved. From the cyclic L-partitions we studied, we observe that each b_i in a period of a bank sequence occurs at most once. If this is true, then Conjecture 5 follows immediately.

By associating each L-partition λ of n with a (0,1)-array, we observe an easy way to describe the cycles in Austrian Solitaire. For example, for $n = 13$, the cycle starting with the cyclic L-partition $\lambda = (0; 7, 4, 1)$ corresponds to the pattern (3, 2, 2, 3, 2, 2, …); for $n = 15$, the cycle starting with the cyclic L-partition $\lambda = (0; 7, 5, 3)$ corresponds to the pattern (3, 3, 3, 3, …). In general, we see some patterns of cycles in Austrian Solitaire. But we have not found an exact formula to describe the cycles so far.

四、 參考文獻

1. **E. Akin and M. Davis**, Bulgarian solitaire, *Amer. Math. Monthly* **4** {1985}, 237--250.

五、 計畫成果自評

We take Conjecture 5 as a corollary of Conjecture 4. However, we found a counterexample of Conjecture 4. We will try other ways to prove Conjecture 5. If this can be done, we will try to submit the results, including the patterns of cycles we found in this project to a journal for publication.