行政院國家科學委員會專題研究計畫　成果報告

奧地利單人牌戲

計畫類別：個別型計畫
計畫編號：NSC91-2115-M-034-001-
執行期間：91 年 08 月 01 日至 92 年 07 月 31 日
執行單位：中國文化大學應用數學系

計畫主持人：何志昌

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中　華　民　國 92 年 10 月 31 日

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奧地利單人牌戲

Austrian Solitaire

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一、 中文摘要

Akin 和 Davis 提出了奧地利單人牌戲。此遊戲是這樣進行的：先將一疊 n 張紙牌分成幾堆，使得每堆的紙牌張數不大於 L，其中 L 是某一個固定的整數。將其中某一特殊堆稱作銀行，保留在旁邊。此遊戲的每次移動包含兩個步驟：首先，從每一普通堆中取出一張牌存入銀行；然後，從銀行裏不斷地分出若干堆，每一堆恰好是 L 張紙牌，直到銀行裏小於 L 張為止 (也有可能銀行裏的紙牌剛好全部被取出)。

給定整數 L 和任意一組已經分成若干堆的 n 張紙牌，我們依照上述規則重複地移動，最後一定會形成一個循環。例如：從 (0;4,3) 開始，其中括號裏的第一個數字代表銀行，重複地移動後，依序得到 (2;3,2), (0;4,2,1), (3;3,1), (1;4,2), (3;3,1) ・・・，形成了(3;3,1), (1;4,2), (3;3,1) , (1;4,2)・・・的循環。

對於奧地利單人牌戲，Akin 和 Davis 提出問題：如何描述這遊戲中形成的循環。他們也猜測：給定紙牌張數 n 和整數 L，循環的個數是 1.在此計劃中，我們將研究奧地利單人牌戲中所形成的循環的一些相關問題。

關鍵詞： 奧地利單人牌戲，循環

二、 英文摘要

Akin and Davis introduced Austrian Solitaire. It proceeds as follows: Start by laying out piles from a deck of n cards such that each pile has size $\leq L$, where L is some fixed integer. One special pile called the bank is reserved on the side. A move of this game consists of two steps. First, remove one card from each ordinary pile and put it in the bank. Now from the bank lay out new piles of size exactly equal to L, continuing until the size of the bank is $< L$ (including the possibility of exhausting the bank). The operation is repeated over and over.

Starting from any division into piles, one always reaches some cycle of partitions of n. For example, starting from (0;4,3), where the first number in the parentheses corresponds to the bank, the successors are (2;3,2), (0;4,2,1), (3;3,1), (1;4,2), (3;3,1)…. Note that (3;3,1), (1;4,2), (3;3,1) , (1;4,2)・・・form a cycle.

Akin and Davis suggested the problem of describing the cycles in Austrian Solitaire. In particular, they conjectured that for any fixed deck size n and fixed integer L, there is a unique cycle. We will study the cycles of Austrian Solitaire in this project.

Keywords: Austrian Solitaire, cycle

三、 報告內容

(一) 前言及研究目的：

Akin and Davis [1] introduced *Austrian Solitaire*. (The idea for the game arose when they were reading a discussion of the so-called Austrian school of capital theory.) It proceeds as follows: Start by laying out piles from a deck of n cards such that each pile has size $\leqq L$, where L is some fixed integer. One special pile called the *bank* is reserved on the side.

A move A_L of this game consists of two steps. First, remove one card from each ordinary pile and put it in the bank. Now from the bank lay out new piles of size exactly equal to L, continuing until the size of the bank is $< L$ (including the possibility of exhausting the bank). The operation A_L is repeated over and over.

Assume n and L are positive integers and $n \geq L$. We say $\lambda = (\lambda_0; \lambda_1, \lambda_2, \ldots, \lambda_s)$ is an L-partition of n (with s parts), provided $0 \leq \lambda_0 < L$, $L \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 1$, and these integers λ_i add up to n. Then each λ corresponds to a stage in Austrian Solitaire with the bank λ_0 and ordinary piles λ_i, $1 \leq i \leq s$. So Austrian Solitaire can be viewed as a way of changing one L-partition of n into another. For instance, if $L = 4$ and $\lambda = (0;4,3)$, then the successors of λ are $A_L(\lambda) = (2;3,2)$, $A_L^{(2)}(\lambda) = (0;4,2,1)$, $A_L^{(3)}(\lambda) = (3;3,1)$, $A_L^{(4)}(\lambda) = (1;4,2)$, $A_L^{(5)}(\lambda) = (3;3,1)$,…. Note that (3;3,1), (1;4,2), (3;3,1), (1;4,2), … form a cycle.

This game has the following economic interpretation: Think of the ordinary piles as machines. Each machine has, when new, a life of exactly L years. The size of a pile is the number of productive years left for a particular machine. Each year it ages one year (and so one card is removed from the pile). For each machine on line the company deposits $1/L$ of its cost into the bank as a sinking fund. Then it buys as many new machines as it can afford, and the remaining funds are left in the bank until next year.

The purpose of this project is to study the cycles in Austrian Solitaire.

(二) 文獻探討及研究方法

For Austrian Solitaire, Akin and Davis[1] proposed

Conjecture 1. *For any fixed deck size n and fixed integer L, there is a unique cycle.*

We say an L-partition of n is *cyclic* if $A_L^{(i)}(\lambda) = \lambda$ for some $i > 0$.

Note that $\lambda = (0; 5, 2)$ is a cyclic 5-partition of 7 and $\lambda' = (0;5,5,4,3,2,2,1)$ is a cyclic 5-partition of 22. In general, we have

Proposition 2. *Let* $n' = n + (1+2+ \ldots + L)$.

(1) If λ is a cyclic L-partition of n, and λ' is the partition obtained by adding the L parts

$\{L, L-1, \ldots, 1\}$ *to* λ, *then* λ' *is a cyclic L-partition of* n'.

(2) *Let* λ' *be a cyclic L-partition of* n'. *If every L-partition in the cycle of* λ' *consists of at least one part of i, for* $i = 1, \ldots, L$, *and* λ *is the partition obtained by removing the L parts* $\{L, L-1, \ldots, 1\}$ *from* λ', *then* λ *is a cyclic L-partition of n.*

Conjecture 3. *If* $n' > 1+2+ \ldots + L$, *then every cyclic L-partition of* n' *consists of at least one part of i, for* $i = 1, \ldots, L$. *(Thus the number of cycles for* n' *cards equals the number of cycles for* $n - (1+2+ \ldots + L)$ *cards.)*

If Conjecture 3 is true, without loss of generality, we may assume that the number of cards in Austrian solitaire is at most $1+2+ \ldots + L$.

Note that $\lambda = (0;\ 5,\ 2)$ is a cyclic 5-partition of 7 and $\lambda' = (4;4,3,1)$ is a cyclic 5-partition of 12. In general, we have

Proposition 4. *Let* $n + n' = 1+2+ \ldots + L + (L-1)$. *Let* $\lambda = (\lambda_0;\ \lambda_1, \lambda_2, \ldots, \lambda_s)$ *be a cyclic L-partition of n. If* $\lambda_i \neq \lambda_j$ *for all* $i > j > 0$, *then the L-partition of* n'

$$\lambda' = (L-1-\lambda_0;\ \lambda'_1,\ \lambda'_2, \ldots,\ \lambda'_{L-s}),$$

where $\{\lambda'_1, \lambda'_2, \ldots, \lambda'_{L-s}\} = \{1, 2, \ldots, L\} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$, *is cyclic.*

The condition $\lambda_i \neq \lambda_j$ in Proposition 4 seems not necessary, we propose

Conjecture 5. *Let* $n' < 1+2+ \ldots + L + (L-1)$. *If* $\lambda = (\lambda_0;\ \lambda_1, \lambda_2, \ldots, \lambda_s)$ *is a cyclic L-partition of n, then* $\lambda_i \neq \lambda_j$ *for all* $i > j > 0$.

For a cyclic *L*-partition λ of *n*, the *period* of λ is the smallest positive integer *k* satisfying $A_L^{(k)}(\lambda) = \lambda$. For example, when $L = 5$ the period of $\lambda = (0;\ 5,\ 2)$ is 3. We propose

Conjecture 6. *For any cyclic L-partition* λ *of n, the period of* λ *is at most L.*

The goal of this project is to prove or disprove some of the conjectures listed above.

(三) 結果與討論：

By associating each *L*-partition λ of *n* with a (0,1)-array, we showed some properties of Austrian solitaire and we proved both Conjectures 3 and 5. Thus, without loss of generality, we may assume that the number of cards in Austrian solitaire is at most $1+2+ \ldots + L$. Further, if $n + n' = 1+2+ \ldots + L + (L-1)$, there is a one to one correspondence between the cyclic L-partitions of n and the cyclic L-partitions of n'.

四、 參考文獻

1. **E. Akin and M. Davis**, Bulgarian solitaire, *Amer. Math. Monthly* **4** {1985}, 237--250.

五、 計畫成果自評

The proofs of Conjectures 3 and 5 are found. Though we have not solved Conjecture 1 proposed by Akin and Davis [1]. But the results of this project cut down the details of Austrian solitaire and can help us learn the cycles in Austrian solitaire. We will try to submit the results to a journal for publication.