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Optimal Profit for Fuzzy Demand in The Fuzzy Sense

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ABSTRACT

The linear relationship of the crisp demand function is represented by $p = a + bx$, $0 \le x \le \frac{a}{-b}$, $a > 0$, $b < 0$. When the crisp revenue function is $R(x) = a + bx^2$ and crisp cost function as $\pi(x) = u + vx$, the profit function is $N(x) = ax + bx^2 - u - vx$. The demand quantity with respect to price p is $x\,(= {}^{(p-a)}\!/_{b})$. In a perfect competition market, there can be slight fluctutation and for the demand quantity d at price p, the points of demand quantities in a straight line vary in the interval between $x - \Delta$ and $x + \Delta$ ($0 < \Delta < x$, in which Δ is to be selected), it means that the fuzzification of d is $\widetilde{D}$. For demand quantity d, the profit function is $N(d) = ad + bd^2 - u - vd$. If $bd^2 + (a - v)d - u = z$, the membership function $\mu_{N(\widetilde{D})}(z)$ and its centroid $M(\Delta, x)$ of fuzzy profit $N(\widetilde{D})$ can be obtained by the extension principle. This centroid value is the estimate value of the profit for demand quantity x in the fuzzy sense. By numerical method, the maximum profit for fuzzy number $(x^{**} - \Delta^{**}, x^{**} + \Delta^{**})$ is $\max\limits_{0<\Delta<x\le\frac{a}{-b}} M(\Delta^{**}, x^{**})$. If actual fuzzy situation is not $(x^{**} - \Delta^{**}, x^{**} + \Delta^{**})$, the estimate value of profit is not good, then the optimal solution of crisp function can be compared with the maximum profit $M(\Delta^{**}, x^{**})$ for the optimum quantity of demand x^{**} in fuzzy sense.

Scope and Purpose

The demand function is $p = a + bx$, $0 \le x \le \frac{a}{-b}$, $a > 0$, $b < 0$ and cost function is $\pi(x) = u + vx$, $0 \le x$, $u > 0$, $v > 0$. In monoply market, the profit function is $N(x) = ax + bx^2 - u - vx$, $0 \le x \le \frac{a}{-b}$. From $N(x)$, we find when the demand x_* is $\frac{a-v}{2(-b)}$, $N(x_*)$ is the maximum profit. But in perfect competition market, with the same price p, the demand is not $x\left(= \frac{p-a}{b}\right)$ in general. It may fluctuate around x. We can fuzzify the demand quantity to a fuzzy number $\widetilde{D} = (x - \Delta, x, x + \Delta)$, $0 < \Delta < x$. In this way, we get fuzzy profit $N(\widetilde{D})$. Through extension principle, we can find its membership function $\mu_{N(\widetilde{D})}(z)$ and its centroid $M(\Delta, x)$. This centroid can be used as an estimate of profit when demand is x in the fuzzy sense. If $\Delta \in (0, x)$, then we can use numerical approximation from $M(\Delta, x)$ to find x^{**}

such that $M(\Delta, x^{**})$ is the maximum.

KEYWORDS: fuzzy profit, fuzzy demand, extension principle.

1. INTRODUCTION

In [2], he considered the compound interest in the fuzzy sense. In [7], he considered a fuzzy model of economic choice. In [6] he considered the profit apportionment in mutually owned concerns and their expected profit in the fuzzy sense. Yao and the others [3, 8~10] considered economic principle in the fuzzy sense, consumer surplus and producer surplus in the fuzzy sense and best prices of two mutually complementary merchandise in the fuzzy sense. These were the economical problems using fuzzy concepts to consider in the past. In this article, we treat the profit for fuzzy demand in the fuzzy sense.

The linear relationship of the crisp demand function is represented by $P = a + bx$, $0 \leq x \leq \frac{a}{-b}$, $a > 0$, b<0 when the crisp revenue function is $R(x) = a + bx^2$, $0 \leq x \leq \frac{a}{-b}$ and crisp cost function as $\pi(x) = u + vx$, $x \geq 0$, the profit function is $N(x) = ax + bx^2 - u - vx$. In a perfect competition market for the same price p, the daily demand quantity is not fixed there can be slight fluctuation around the point $x\left(= {(p-a)}/{b}\right)$ in the straight line, for which the points of demand quantities in this straight line vary in the interval between $x - \Delta$ and $x + \Delta$, $0 < \Delta < x$. By fuzzification of the demand quantity d as $\widetilde{D}$, at price p with d varying around the point $x\left(= {(p-a)}/{b}\right)$, then the membership function fuzzy demand $\widetilde{D}$ is assumed to be fuzzy number $\widetilde{D} = (x - \Delta, x, x + \Delta)$
the centroid of $\mu_{\widetilde{D}}(d)$, $\mu_0 = x$ is corresponding to the estimated value of demand quantity in fuzzy sense at price p, the centroid is identical with crisp demand $x\left(= {(p-a)}/{b}\right)$ at price p.

Assume the profit function for demand d is $N(d) = bd^2 + (a - v)d - u$, from $N(d) = bd^2 + (a - v)d - u = z$, by extension principle the membership function $\mu_{N(\widetilde{D})}(d)$ of the fuzzy profit function $N(\widetilde{D})$. When $x_* = \frac{a - v}{2(-b)}$, $N(x_*) = -u + \frac{(a - v)^2}{4(-b)}$ is the maxima. Assume $x_1 = x - \Delta$, $x_2 = x$, $x_3 = x + \Delta$, for finding $\mu_{N(\widetilde{D})}(z)$ conveniently, $0 < x_1 < x_2 < x_3$ divide into 4 different condition for calculation; in §2.1 condition $x_* < x_1 < x_2 < x_3$ find $\mu_{N(\widetilde{D})}(z)$ and centroid $M_1(\Delta, x)$, in §2.2 condition $0 < x_1 < x_2 < x_3 < x_*$ find $\mu_{N(\widetilde{D})}(z)$ and centroid $M_2(\Delta, x)$, in §2.3 condition $0 < x_1 < x_* < x_2 < x_3$ find $\mu_{N(\widetilde{D})}(z)$ and centroid $M_3(\Delta, x)$, in §2.4 condition $0 < x_1 < x_2 < x_* < x_3$ find $\mu_{N(\widetilde{D})}(z)$ and centroid

$M_4(\Delta, x)$. In §3, from §2.1~§2.4 the centroid $M(\Delta, x)$ of $\mu_{N(\tilde{D})}(z)$ and the maxima point (Δ^{**}, x^{**}) of $M(\Delta, x)$ are calculated by approximate solution, i.e., we can find the optimal demand quantity x^{**} and the maxima profit $M(\Delta^{**}, x^{**})$ in the fuzzy sense.

In §4, we give examples to find Δ^{**}, x^{**}, $M(\Delta^{**}, x^{**})$ are compared with the optimal quantity x_* and the maxima profit $N(x_*)$ in crisp case.

2. FUZZY PROFIT AND ITS MEMBERSHIP FUNCTION

The linear relationship of the crisp demand function is expressed as

$$p = a + bx, \quad 0 \le x \le \frac{a}{-b} \tag{1}$$

in which $a > 0$, $b < 0$, p is price and x demand quantity, a, b are known constants. The crisp revenue function is

$$R(x) = x \cdot p = ax + bx^2, \quad 0 \le x \le \frac{a}{-b}. \tag{2}$$

Crisp cost function is

$$\pi(x) = u + vx, \quad 0 \le x \tag{3}$$

in which $u > 0$, $0 < v < a$, u, v are known constants.
The crisp profit function is

$$N(x) = R(x) - \pi(x) = -u + (a - v)x + bx^2, \quad 0 \le x \le \frac{a}{-b} \tag{4}$$

when $x = \frac{a - v}{2(-b)} (\equiv x_* \text{ say})$, $N(x_*) = -u + \frac{(a-v)^2}{4(-b)}$ is the maxima value. In actual problem (1) for same price p, the daily demand is not fixed and varies slightly (cf. Fig 1).

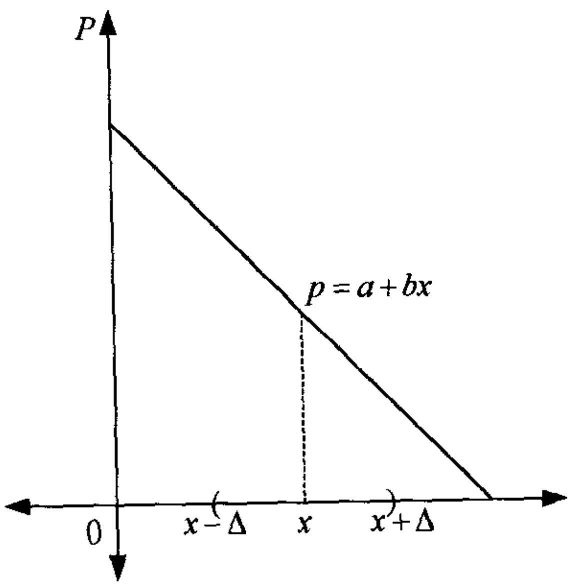


Fig 1. The fluctuation of x

Thus, the problem in fuzzy sense is to consider the fluctuation of $x(= {}^{(p-a)}\!/_{b})$ in the neighborhood of $x - \Delta$ and $x + \Delta$ $(0 < x < \Delta)$(cf. Fig1) for demand quantity d at price p. Since Δ is known in most cases as $0 < \Delta < x$, the fuzzy situations are numerous, however, we would like to find out the maxima value of profit. In crisp case $d = x$, the demand quantity d is related to (1) for fuzzification to $\widetilde{D}$ with following fuzzy number its membership function with respect to any fixed point in the straight line (1) for $x\left(= {}^{(p-a)}\!/\!_{b}\right)$(c.f. Fig.1) is

$$\mu_{\widetilde{D}}(d) = \begin{cases} \dfrac{d - x + \Delta}{\Delta} & , \quad x - \Delta \le d \le x \\ \dfrac{x + \Delta - d}{\Delta} & , \quad x \le d \le x + \Delta \\ 0 & , \quad otherwise \end{cases} \tag{5}$$

in which $0 < \Delta < x \le \dfrac{a}{-b}$, Δ is an unknown to be decided by decision maker, with establishment of $0 < x - \Delta < x < x + \Delta$. Here $\widetilde{D}$ is referred as fuzzy demand quantity with fuzzy number of $(x - \Delta, x, x + \Delta)$, which varies with the change of x. The centroid of $\mu_{\widetilde{D}}(d)$ is

$$M_D = \frac{\int_{-\infty}^{\infty} t \cdot \mu_{\widetilde{D}}(t)dt}{\int_{-\infty}^{\infty} \mu_{\widetilde{D}}(t)dt} = x\left(= \frac{p-a}{b}\right).$$

For demand quantity in the fuzzy sense at price p, $M_D = x$ is the estimated value to be identical with demand quantity x in the crisp case. If the revenue function for demand quantity d is $R(d) = ad + bd^2$ and cost $\pi(d) = u + vd$, then profit is $N(d) = R(d) - \pi(d) = -u + (a - v)d + bd^2$, $0 \le d \le \dfrac{a}{-b}$. Assume

$$N(d) = bd^2 + (a - v)d - u = z \tag{6}$$

by extension principle the membership function of the fuzzy set $N(\widetilde{D})$(fuzzy profit), is obtained as following

$$\mu_{N(\widetilde{D})}(z) = \begin{cases} \sup\limits_{d \in N^{-1}(z)} \mu_{\widetilde{D}}(d) & if \quad N^{-1}(z) \ne \phi \\ 0 & if \quad N^{-1}(z) = \phi \end{cases} \tag{7}$$

The determinants of the quadratic equation for d in (6) are

$$D_1 = (a - v)^2 + 4b(u + z) \ge 0 \ \text{ if } \ z \le -u + \frac{(a - v)^2}{4(-b)} \left(= N(x_*)\right).$$

For $D_1 > 0$, the two roots are

$$d_1 = \frac{-(a - v) + \sqrt{D_1}}{2b} = x_* - \frac{\sqrt{D_1}}{2(-b)} \le x_*, \quad d_2 = \frac{-(a - v) - \sqrt{D_1}}{2b} = x_* + \frac{\sqrt{D_1}}{2(-b)} \ge x_*.$$

For $D_1 = 0$, the duplicate root of (6) is $d_1 = d_2 = x_*$. Then,

$$d_1 \le x_* \le d_2, \quad d_1 + d_2 = 2x_* \tag{8}$$

From (7), we obtained

$$\text{when } \ z \le -u + \frac{(a - v)^2}{4(-b)} \left(= N(x_*)\right), \quad \mu_{N(\widetilde{D})}(z) = \max\left[\mu_{\widetilde{D}}(d_1), \mu_{\widetilde{D}}(d_2)\right] \tag{9}$$

when $z > -u + \frac{(a-v)^2}{4(-b)}$, $\mu_{N(\tilde{D})}(z) = 0$ (10)

Let $x_1 = x - \Delta$, $x_2 = \Delta$, $x_3 = x + \Delta$, $0 < \Delta < x \le \frac{a}{-b}$, from (5)

$$\mu_{\tilde{D}}(d_j) = \begin{cases} \frac{d_j - x + \Delta}{\Delta} & , \quad x_1 \le d_j \le x_2 \\ \frac{x + \Delta - d_j}{\Delta} & , \quad x_2 \le d_j \le x_1 \\ 0 & , \quad otherwise \end{cases} \tag{11}$$

In this section, we obtain $\mu_{N(\tilde{D})}(z)$ and its centroid $M(\Delta, x)$, the later being the estimated value of profit for demand quantity x in the fuzzy sense. In (5) the fuzzy number $(x - \Delta, x, x + \Delta)$, x and Δ satisfy the relationship $0 < \Delta < x \le \frac{a}{-b}$ with centroid x. When x and Δ vary between $0 < \Delta < x \le \frac{a}{-b}$, the fuzzy number $(x - \Delta, x, x + \Delta)$ also varies accordingly with respect to real number variable, x. When the define the fuzzy number $(x - \Delta, x, x + \Delta)$ as fuzzy number variable. In crisp sense, the maxima profit, $N(x_*)$ of demand quantity x as real number variable can be found in the optimal demand quantity x_* in the region of $0 < x \le \frac{a}{-b}$. With respect to this situation in the fuzzy sense, the optimal fuzzy number $(x^{**} - \Delta^{**}, x^{**}, x^{**} + \Delta^{**})$ can be found from fuzzy number variable $(x - \Delta, x, x + \Delta)$ in the region of $0 < \Delta < x \le \frac{a}{-b}$ to maximize for the centroid(estimated value of profit) $M(\Delta^{**}, x^{**})$. In §4, we obtained optimal demand quantity x^{**} in the fuzzy sense and maxima profit $M(\Delta^{**}, x^{**})$ are compared with the values of optimal demand quantity x_* and maxima profit $N(x_*)$ in crisp sense respectively; it means that under which condition in the fuzzy sense, the estimated value of profit is at maxima. In the event of actual fuzzy phenomena for fuzzy number $(x^{**} - \Delta^{**}, x^{**}, x^{**} + \Delta^{**})$ with estimated profit value of $M(\Delta^{**}, x^{**})$ as maxima. If the actual fuzzy number $(x - \Delta, x, x + \Delta)$ is not the fuzzy number $(x^{**} - \Delta^{**}, x^{**}, x^{**} + \Delta^{**})$, the estimated profit value of $M(\Delta, x)$ is not at maxima, which can be applied to determine whether maxima profit is obtained.

For finding $\mu_{N(\tilde{D})}(z)$ in (9), we consider the following positions of d_1 and d_2 from (11).

Table 1. The positions of d_1 and d_2

case	$x_1 = x - \Delta$	$x_2 = x$		$x_3 = x + \Delta$	$\mu_{N(\tilde{d})}(z)$
(1)	d_1 d_2				0
(2)	d_1	d_2			$\frac{(d_2 - x + \Delta)}{\Delta}$

(3)	d_1		d_2		$\frac{(x+\Delta-d_2)}{\Delta}$
(4)	d_1			d_2	0
(5)		$d_1 \quad d_2$			$\frac{(d_2-x+\Delta)}{\Delta}$
(6)		d_1	d_2		$\max\left[\frac{d_1-x+\Delta}{\Delta}, \frac{x+\Delta-d_2}{\Delta}\right]$
(7)		d_1		d_2	$\frac{(d_1-x+\Delta)}{\Delta}$
(8)			$d_1 \quad d_2$		$\frac{(x+\Delta-d_1)}{\Delta}$
(9)			d_1	d_2	$\frac{(x+\Delta-d_1)}{\Delta}$
(10)				$d_1 \quad d_2$	0

Since, $\frac{d_1-x+\Delta}{\Delta} \le \frac{x+\Delta-d_2}{\Delta} \Leftrightarrow d_1+d_2 \le 2x \Leftrightarrow x_* \le x$ then

$$\max\left[\frac{d_1-x+\Delta}{\Delta}, \frac{x+\Delta-d_2}{\Delta}\right] = \begin{cases} \frac{d_1-x+\Delta}{\Delta} & if \quad 0 \le x \le x_* \\ \frac{x+\Delta-d_2}{\Delta} & if \quad x_* \le x \end{cases} \tag{12}$$

The following relationship is applied to obtain $\mu_{N(\tilde{D})}(z)$, assuming

$e_j = -u + (a-v)x_j + bx_j^2, \quad j = 1,2,3.$

$-u + (a-v)x_j + bx_j^2 - \left[-u + \frac{(a-v)^2}{4(-b)}\right] = \frac{1}{4b}\left[a - v + 2bx_j\right]^2 \le 0$ (Since $b < 0$), then

$$e_j = -u + (a-v)x_j + bx_j^2 \le -u + \frac{(a-v)^2}{4(-b)}, \quad j = 1,2,3 \tag{13}$$

$e_i - e_j = b(x_i - x_j)\left[x_i + x_j - \frac{(a-v)}{-b}\right] = b(x_i - x_j)\cdot\left[x_i + x_j - 2x_*\right].$

Since $b < 0$, $i < j$, $x_i < x_j$, then

$$\text{for } i < j,\ e_i - e_j = \begin{cases} > 0 & if \quad x_i + x_j > 2x_* \\ < 0 & if \quad x_i + x_j < 2x_* \\ = 0 & if \quad x_i + x_j = 2x_* \end{cases} \tag{14}$$

From (9) the follow range of z should satisfy $z \le -u + \frac{(a-v)^2}{4(-b)}$

(1°) $x_j \le d_1$, $j = 1,2,3$, $x_j \le \frac{a - v - \sqrt{D_1}}{2(-b)} \Leftrightarrow \sqrt{D_1} \le a - v - 2(-b)x_j$

then if $a - v - 2(-b)x_j \ge 0$, i.e. when $0 < x_j \le x_*$, thus by squaring both sides of the above inequality, we obtain $e_j \le z$, then from (13)

$$(x_j \le d_1) \wedge (0 < x_j \le x_*) \Rightarrow e_j \le z \le -u + \frac{(a-v)^2}{4(-b)} \tag{15}$$

(2°) $d_1 \le x_j$, $j = 1,2,3 \Leftrightarrow \sqrt{D_1} \ge a - v - 2(-b)x_j$, we obtain the following two sets of inequalities, from (13)

$$\begin{cases} 0 < x_j \le x_* \\ z \le e_j \end{cases} \quad \text{or} \quad \begin{cases} d^* \le x_j \\ z \le -u + \dfrac{(a-v)^2}{4(-b)} \end{cases}$$

$$(d_1 \le x_j) \wedge (0 < x_j \le x_*) \Rightarrow z \le e_j \tag{16}$$

$$(d_1 \le x_j) \wedge (x_* \le x_j) \Rightarrow z \le -u + \frac{(a-v)^2}{4(-b)} \tag{17}$$

(3°) $x_j \le d_2$, $j = 1,2,3$, $x_j \le \dfrac{a - v + \sqrt{D_1}}{2(-b)} \Leftrightarrow 2(-b)x_j - (a-v) \le \sqrt{D_1}$, we obtain the following

$$(x_j \le d_2) \wedge (0 < x_j \le x_*) \Rightarrow z \le -u + \frac{(a-v)^2}{4(-b)} \tag{18}$$

$$(x_j \le d_j) \wedge (x_* \le x_j) \Rightarrow z \le e_j \tag{19}$$

(4°) $d_2 \le x_j$, $j = 1,2,3 \Leftrightarrow 2(-b)x_j - (a-v) \ge \sqrt{D_1}$, we obtain the following

$$(d_2 \le x_j) \wedge (x_* \le x_j) \Rightarrow e_j \le z \le -u + \frac{(a-v)^2}{4(-b)} \tag{20}$$

For finding the centroid of $\mu_{N(\tilde{a})}(z)$, we need the following

Let

$$\begin{aligned} & G_1(q_1, q_2) \\ &= \int_{q_1}^{q_2} \sqrt{(a-v)^2 + 4b(u+z)}\,dz \\ &= \frac{1}{6b}\left\{\left[(a-v)^2 + 4bu + 4bq_2\right]^{3/2} - \left[(a-v)^2 + 4bu + 4bq_1\right]^{3/2}\right\} \end{aligned}$$

$$\begin{aligned} & G_2(q_1, q_2) = \int_{q_1}^{q_2} z\sqrt{(a-v)^2 + 4b(u+z)}\,dz \\ &= \frac{1}{40b^2}\left\{\left[(a-v)^2 + 4b(u+q_2)\right]^{5/2} - \left[(a-v)^2 + 4b(u+q_1)\right]^{5/2}\right\} \\ & \quad - \frac{(a-v)^2 + 4bu}{24b^2}\left\{\left[(a-v)^2 + 4b(u+q_2)\right]^{3/2} - \left[(a-v)^2 + 4b(u+q_1)\right]^{3/2}\right\} \end{aligned}$$

$$\begin{aligned} V_1(q_1, q_2) &= \int_{q_1}^{q_2} d_1 \cdot dz = \int_{q_1}^{q_2} \frac{-(a-v) + \sqrt{(a-v)^2 + 4b(u+z)}}{2b}\,dz \\ &= \frac{-a+v}{2b}(q_2 - q_1) + \frac{1}{2b}G_1(q_1, q_2) \end{aligned} \tag{21}$$

$$\begin{aligned} V_2(q_1, q_2) &= \int_{q_1}^{q_2} d_2 \cdot dz = \int_{q_1}^{q_2} \frac{-(a-v) - \sqrt{(a-v)^2 + 4b(u+z)}}{2b}\,dz \\ &= \frac{-a+v}{2b}(q_2 - q_1) - \frac{1}{2b}G_1(q_1, q_2) \end{aligned} \tag{22}$$

$$V_{12}(q_1, q_2) = \int_{q_1}^{q_2} z \cdot d_1 \cdot dz = \frac{-a+v}{4b}(q_2^2 - q_1^2) + \frac{1}{4b}G_2(q_1, q_2) \tag{23}$$

$$V_{22}(q_1, q_2) = \int_{q_1}^{q_2} z \cdot d_2 \cdot dz = \frac{-a+v}{4b}(q_2^2 - q_1^2) - \frac{1}{2b}G_2(q_1, q_2) \tag{24}$$

To find conveniently $\mu_{N(\tilde{a})}(z)$ and its centroid, we devide $0 < x_1 < x_3 < x_3$ into four different conditions:

(1) $x_* < x_1 < x_3 < x_3$ (2) $0 < x_1 < x_2 < x_3 < x_*$

(3) $0 < x_1 < x_* < x_2 < x_3$ (4) $0 < x_1 < x_2 < x_* < x_3$

2.1 $\mu_{N(\tilde{a})}(z)$ AND ITS CENTROID UNDER THE CONDITION $x_* < x_1 < x_2 < x_3$

In condition, $x_* < x_1 < x_2 < x_3$ (25)

from (14), we have

$$e_3 < e_2 < e_1 \quad (26)$$

From table 1, (13)~(20) and (25), (26), we have the following:

(1)case (1), (4), (10), $\mu_{N(\tilde{a})}(z) = 0$.

(2)case(5)~(9), there are one inequality $x_j \le d_1$, $j = 1,2$ and $x_* < x_j$;then, from (1°) and (15), no solution.

(3)case(2) $d_1 \le x_1 \le d_2 \le x_2$, since $x_* < x_j$, $j = 1,2,3$, then

$(d_1 \le x_1) \wedge (x_* < x_1)$, from (17), then $z \le -u + \dfrac{(a-v)^2}{4(-b)}$ and

$(x_1 \le d_2) \wedge (x_* < x_1)$, from (19), then $z \le e_1$ and

$(d_2 < x_2) \wedge (x_* < x_2)$, from (20), then $e_2 \le z \le -u + \dfrac{(a-v)^2}{4(-b)}$. From (13) and (26), we have

$$e_2 \le z \le e_1 \quad \text{and} \quad \mu_{N(\tilde{a})}(z) = \frac{d_2 - x + \Delta}{\Delta} \quad (27)$$

(4)case(3) $d_1 \le x_1$ and $x_2 \le d_2 \le x_3$. Since $x_* < x_j$, $j = 1,2,3$, from (17), (19), (20), we have $z \le -u + \dfrac{(a-v)^2}{4(-b)}$, $z \le e_2$ and $e_3 \le z \le -u + \dfrac{(a-v)^2}{4(-b)}$, from (13) and

(26), we have $e_3 \le z \le e_2$ and $\mu_{N(\tilde{a})}(z) = \dfrac{x + \Delta - d_2}{\Delta}$ (28)

From (27), (28), we have the follow property 2.1.

Since $x_1 = x - \Delta$, $x = x_2$, $x_3 = x + \Delta$, $0 < \Delta < x \le \dfrac{a}{-b}$, therefore $0 < x_1 < x_2 < x_3$ is valid and $x_* < x_1 < x_2 < x_3$ can be converted into a condition $x_* < x - \Delta$ and $0 < \Delta < x \le \dfrac{a}{-b}$.

Property 2.1 For the condition $x_* < x - \Delta$ and $0 < \Delta < x \le \dfrac{a}{-b}$, the membership function of $N(\tilde{D})$ is

$$
\mu_{N(\tilde{D})}(z)=\begin{cases}\dfrac{x+\Delta-d_2}{\Delta} & , \quad e_3 \le z \le e_2 \\ \dfrac{d_2-x+\Delta}{\Delta} & , \quad e_2 \le z \le e_1 \\ 0 & , \quad otherwise\end{cases} \tag{29}
$$

Property 2.2 For the condition $x_* < x-\Delta$ and $0<\Delta<x\le\dfrac{a}{-b}$, the centroid of (29) of $\mu_{N(\tilde{a})}(z)$ is

$$
M_1(\Delta, x)=\frac{R_1(\Delta, x)}{P_1(\Delta, x)} \tag{30}
$$

At here, from (21)~(24), we have

$$
P_1(\Delta, x)=\int_{-\infty}^{\infty}\mu_{N(\tilde{D})}(z)dz
$$

$$
=\frac{x+\Delta}{\Delta}(e_2-e_3)-\frac{1}{\Delta}V_2(e_3,e_2)+\frac{1}{\Delta}V_2(e_2,e_1)+\frac{-x+\Delta}{\Delta}(e_1-e_2)
$$

$$
R_2(\Delta, x)=\int_{-\infty}^{\infty}z\cdot\mu_{N(\tilde{D})}(z)dz
$$

$$
=\frac{x+\Delta}{2\Delta}\left(e_2^2-e_3^2\right)-\frac{1}{\Delta}V_{22}(e_3,e_2)+\frac{1}{\Delta}V_{22}(e_2,e_1)+\frac{-x+\Delta}{2\Delta}\left(e_1^2-e_2^2\right)
$$

2.2 $\mu_{N(\tilde{a})}(z)$ AND ITS CENTROID UNDER THE CONDITION $0<x_1<x_2<x_3<x_*$

In condition, $0<x_1<x_2<x_3<x_*$ (31)

from (14), we have

$e_1<e_2<e_3$ (32)

From table 1, (13)~(20) and (31), (32), similarly section 2.1, we have the following:

Property 2.3 For the condition $x+\Delta<x_*$ and $0<\Delta<x\le\dfrac{a}{-b}$, the membership function of $N(\tilde{D})$ is

$$
\mu_{N(\tilde{D})}(z)=\begin{cases}\dfrac{d_1-x+\Delta}{\Delta} & , \quad e_1 \le z \le e_2 \\ \dfrac{x+\Delta-d_1}{\Delta} & , \quad e_2 \le z \le e_3 \\ 0 & , \quad otherwise\end{cases} \tag{35}
$$

Property 2.4 For the condition $x+\Delta<x_*$ and $0<\Delta<x\le\dfrac{a}{-b}$, the centroid of (35) of $\mu_{N(\tilde{D})}(z)$ is

$$
M_2(\Delta, x)=\frac{R_2(\Delta, x)}{P_2(\Delta, x)}. \tag{36}
$$

At here, from (21)~(24), we have

$$P_2(\Delta, x)=\int_{-\infty}^{\infty}\mu_{N(\tilde{D})}(z)dz$$
$$=\frac{1}{\Delta}V_1(e_1,e_2)+\frac{-x+\Delta}{\Delta}(e_2-e_1)+\frac{x+\Delta}{\Delta}(e_3-e_2)-\frac{1}{\Delta}V_1(e_2,e_3)$$
$$R_2(\Delta, x)=\int_{-\infty}^{\infty}z\cdot\mu_{N(\tilde{D})}(z)dz$$
$$=\frac{1}{\Delta}V_{12}(e_1,e_2)+\frac{-x+\Delta}{2\Delta}(e_2^2-e_1^2)+\frac{x+\Delta}{2\Delta}(e_3^2-e_2^2)-\frac{1}{\Delta}V_{12}(e_2,e_3).$$

2.3 $\mu_{N(\tilde{d})}(z)$ AND ITS CENTROID UNDER THE CONDITION $0<x_1<x_*<x_2<x_3$

In condition, $0<x_1<x_*<x_2<x_3$ (37)

from (14), we have

$e_3<e_2$. (38)

Since $x_1=x-\Delta$, $x_2=x$, $x_3=x+\Delta$, $0<\Delta<x\leq\frac{a}{-b}$, then $x_1+x_3=2x$, $x_1+x_2=2x-\Delta$, then $x_1+x_3<2x_*\Leftrightarrow 0<x<x_*$, $x_1+x_2>2x_*\Leftrightarrow x>x_*+\frac{\Delta}{2}$,

$x_1+x_3>2x_*$ and $x_1+x_2<2x_*\Leftrightarrow x_*<x<x_*+\frac{\Delta}{2}$ (39)

e_1, e_2, e_3, in condition (38), (39), from (14), we have the following permutation

$e_1<e_3<e_2$ if $0<x<x_*(\Leftrightarrow x_1+x_3<2x_*)$ (40)

$e_3<e_1<e_2$ if $x_*<x<x_*+\frac{\Delta}{2}(\Leftrightarrow x_1+x_3>2x_*$ and $x_1+x_2<2x_*)$ (41)

$e_3<e_2<e_1$ if $x_*+\frac{\Delta}{2}<x(\Leftrightarrow$ since $x_1+x_2>2x_*)$ (42)

From table 1, (13)~(20) and (37)~(42), we obtain the following:

(1)case (1), (4), (10), $\mu_{N(\tilde{d})}(z)=0$.

(2)case (8), (9) have one inequality $x_2\leq d_1$ and $x_*<x_2$, from (1°), (15) no solution.

(3)case(2), $d_1\leq x_1\leq d_2\leq x_2$, since $x_1<x_*\leq x_j$, $j=2,3$, then, from (16), (18), (20), we have $z\leq e_1$, $z\leq -u+\frac{(a-v)^2}{4(-b)}$ and $e_2\leq z\leq -u+\frac{(a-v)^2}{4(-b)}$; from (42) and (13), we obtain

when $x_*+\frac{\Delta}{2}<x$, then $\mu_{N(\tilde{D})}(z)=\frac{d_2-x+\Delta}{\Delta}$ if $e_2\leq z\leq e_1$ (43)

(4)case(3) $d_1\leq x_1$ and $x_2\leq d_2\leq x_3$, from (16), (19), (20), we have $z\leq e_1$, $z\leq e_2$ and $e_3\leq z\leq -u+\frac{(a-v)^2}{4(-b)}$; from (41), (42) and (13), we have

when $x_*<x<x_*+\frac{\Delta}{2}$, then $\mu_{N(\tilde{D})}(z)=\frac{x+\Delta-d_2}{\Delta}$ if $e_3\leq z\leq e_1$ (44)

when $x_*+\frac{\Delta}{2}<x$, then $\mu_{N(\tilde{D})}(z)=\frac{x+\Delta-d_2}{\Delta}$ if $e_3\leq z\leq e_2$ (45)

(5)case(5) $x_1 \le d_1$ and $d_2 \le x_2$, from (15), (20), we have

$e_1 \le z \le -u + \frac{(a-v)^2}{4(-b)}$ and $e_2 \le z \le -u + \frac{(a-v)^2}{4(-b)}$; from (40)~(42) and (13), we have

$$\text{when } 0 < x < x_*, \text{ then } \mu_{N(\tilde{D})}(z) = \frac{d_2 - x + \Delta}{\Delta} \text{ if } e_2 \le z \le -u + \frac{(a-v)^2}{4(-b)} \tag{46}$$

$$\text{when } x_* < x < x_* + \frac{\Delta}{2}, \text{ then } \mu_{N(\tilde{D})}(z) = \frac{d_2 - x + \Delta}{\Delta} \text{ if } e_2 \le z \le -u + \frac{(a-v)^2}{4(-b)} \tag{47}$$

$$\text{when } x_* + \frac{\Delta}{2} < x, \text{ then } \mu_{N(\tilde{D})}(z) = \frac{d_2 - x + \Delta}{\Delta} \text{ if } e_1 \le z \le -u + \frac{(a-v)^2}{4(-b)} \tag{48}$$

(6)case(6) $x_1 \le d_1 \le x_2 \le d_2 \le x_3$, from (15), (17), (19), (20), we have

$e_1 \le z \le -u + \frac{(a-v)^2}{4(-b)}$, $z \le -u + \frac{(a-v)^2}{4(-b)}$, $z \le e_2$ and $e_3 \le z \le -u + \frac{(a-v)^2}{4(-b)}$;

from (40), (41) and (9), (13), we have the following

$$\text{when } 0 < x < x_*, \mu_{N(\tilde{D})}(z) = \max\left[\frac{d_1 - x + \Delta}{\Delta}, \frac{x + \Delta - d_2}{\Delta}\right] \text{ if } e_3 \le z \le e_2 \tag{49}$$

$$\text{when } x_* < x < x_* + \frac{\Delta}{2}, \mu_{N(\tilde{D})}(z) = \max\left[\frac{d_1 - x + \Delta}{\Delta}, \frac{x + \Delta - d_2}{\Delta}\right] \text{ if } e_1 \le z \le e_2 \tag{50}$$

(7)case(7) $x_1 \le d_1 \le x_2$ and $x_3 \le d_2$, from (15), (17), (19), we have

$e_1 \le z \le -u + \frac{(a-v)^2}{4(-b)}$, $z \le -u + \frac{(a-v)^2}{4(-b)}$ and $z \le e_3$;

from (40) and (13), we have

$$\text{when } 0 < x < x_*, \text{ then } \mu_{N(\tilde{D})}(z) = \frac{d_1 - x + \Delta}{\Delta} \text{ if } e_1 \le z \le e_3 \tag{51}$$

Summarizing above, we obtain the following property 2.5.

Since $0 < \Delta < x \le \frac{a}{-b}$ and $0 < x_1 < x_2 < x_3$ is valid, $x_1 = x - \Delta$, $x_2 = x$, $x_3 = x + \Delta$, therefore $0 < x_1 < x_* < x_2 < x_3$ can be converted into $x - \Delta < x_* < x$ and $0 < \Delta < x \le \frac{a}{-b}$.

Property 2.5 For the condition $x - \Delta < x_* < x$ and $0 < \Delta < x \le \frac{a}{-b}$, the membership function of $N(\tilde{D})$ is the following as

(1°)if $0 < x < x_*$, $(e_1 < e_3 < e_2)$ then, from (51), (49), (46), (12), we obtain

$$\mu_{N(\tilde{D})}(z)=\begin{cases}\dfrac{d_1-x+\Delta}{\Delta} &, \quad e_1\le z\le e_2\\ \dfrac{d_2-x+\Delta}{\Delta} &, \quad e_2\le z\le -u+\dfrac{(a-v)^2}{4(-b)}\\ 0 &, \quad otherwise\end{cases} \tag{52}$$

(2°) if $x_*<x<x_*+\dfrac{\Delta}{2}$, $(e_3<e_1<e_2)$ then, from (44), (50), (47), (12), we obtain

$$\mu_{N(\tilde{D})}(z)=\begin{cases}\dfrac{x+\Delta-d_2}{\Delta} &, \quad e_3\le z\le e_2\\ \dfrac{d_2-x+\Delta}{\Delta} &, \quad e_2\le z\le -u+\dfrac{(a-v)^2}{4(-b)}\\ 0 &, \quad otherwise\end{cases} \tag{53}$$

(3°) if $x_*+\dfrac{\Delta}{2}<x$, $(e_3<e_2<e_1)$ then, from (45), (43), (48), we obtain

$$\mu_{N(\tilde{D})}(z)=\begin{cases}\dfrac{x+\Delta-d_2}{\Delta} &, \quad e_3\le z\le e_2\\ \dfrac{d_2-x+\Delta}{\Delta} &, \quad e_2\le z\le -u+\dfrac{(a-v)^2}{4(-b)}\\ 0 &, \quad otherwise\end{cases} \tag{54}$$

Property 2.6 For the condition $x-\Delta<x_*<x$ and $0<\Delta<x\le\dfrac{a}{-b}$, the centroid of (52)~(54) of $\mu_{N(\tilde{D})}(z)$ is

(4°) if $0<x<x_*$, the centroid of (52) of $\mu_{N(\tilde{D})}(z)$ is

$$M_{31}(\Delta,x)=\frac{R_{31}(\Delta,x)}{P_{31}(\Delta,x)}. \tag{55}$$

At here,

$$P_{31}(\Delta,x)=\int_{-\infty}^{\infty}\mu_{N(\tilde{D})}(z)dz$$

$$=\frac{1}{\Delta}V_1(e_1,e_2)+\frac{-x+\Delta}{\Delta}(e_2-e_1)+\frac{1}{\Delta}V_2\left(e_2,-u+\tfrac{(a-v)^2}{4(-b)}\right)+\frac{-x+\Delta}{\Delta}\left(-u+\tfrac{(a-v)^2}{4(-b)}-e_2\right)$$

$$R_{31}(\Delta,x)=\int_{-\infty}^{\infty}z\cdot\mu_{N(\tilde{D})}(z)dz$$

$$=\frac{1}{\Delta}V_{12}(e_1,e_2)+\frac{-x+\Delta}{2\Delta}(e_2^2-e_1^2)+\frac{1}{\Delta}V_{22}\left(e_2,-u+\tfrac{(a-v)^2}{4(-b)}\right)+\frac{-x+\Delta}{\Delta}\left[\left(-u+\tfrac{(a-v)^2}{4(-b)}\right)^2-e_2^2\right]$$

(5°) if $x_*<x<x_*+\dfrac{\Delta}{2}$, the centroid of (53) of $\mu_{N(\tilde{D})}(z)$ is

$$M_{32}(\Delta,x)=\frac{R_{32}(\Delta,x)}{P_{32}(\Delta,x)}. \tag{56}$$

At here,

$$P_{32}(\Delta,x)=\int_{-\infty}^{\infty}\mu_{N(\tilde{D})}(z)dz$$

$$=\frac{x+\Delta}{\Delta}(e_2-e_3)-\frac{1}{\Delta}V_2(e_3,e_2)+\frac{1}{\Delta}V_2\left(e_2,-u+\tfrac{(a-v)^2}{4(-b)}\right)+\frac{-x+\Delta}{\Delta}\left(-u+\tfrac{(a-v)^2}{4(-b)}-e_2\right)$$

$$R_{32}(\Delta,x)=\int_{-\infty}^{\infty}z\cdot\mu_{N(\tilde{D})}(z)dz$$

$$=\frac{x+\Delta}{2\Delta}(e_2^2-e_3^2)-\frac{1}{\Delta}V_{22}(e_3,e_2)+\frac{1}{\Delta}V_{22}\left(e_2,-u+\tfrac{(a-v)^2}{4(-b)}\right)+\frac{-x+\Delta}{2\Delta}\left[\left(-u+\tfrac{(a-v)^2}{4(-b)}\right)^2-e_2^2\right]$$

(6°) if $x_*+\frac{\Delta}{2}<x$, the centroid of (54) of $\mu_{N(\tilde{D})}(z)$ is

$$M_{33}(\Delta,x)=\frac{R_{33}(\Delta,x)}{P_{33}(\Delta,x)}. \tag{57}$$

At here,

$$P_{33}(\Delta,x)=\int_{-\infty}^{\infty}\mu_{N(\tilde{D})}(z)dz$$

$$=\frac{x+\Delta}{\Delta}(e_2-e_3)-\frac{1}{\Delta}V_2(e_3,e_2)+\frac{1}{\Delta}V_2\left(e_2,-u+\tfrac{(a-v)^2}{4(-b)}\right)+\frac{-x+\Delta}{\Delta}\left[\left(-u+\tfrac{(a-v)^2}{4(-b)}\right)^2-e_2^2\right]$$

$$R_{33}(\Delta,x)=\int_{-\infty}^{\infty}z\cdot\mu_{N(\tilde{D})}(z)dz$$

$$=\frac{x+\Delta}{2\Delta}(e_2^2-e_3^2)-\frac{1}{\Delta}V_{22}(e_3,e_2)+\frac{1}{\Delta}V_{22}\left(e_2,-u+\tfrac{(a-v)^2}{4(-b)}\right)+\frac{-x+\Delta}{2\Delta}\left[\left(-u+\tfrac{(a-v)^2}{4(-b)}\right)^2-e_2^2\right]$$

Assume, $Q_1=\{(\Delta,x)\mid 0<x<x_*\}$

$$Q_2=\left\{(\Delta,x)\,\middle|\,x_*<x<x_*+\frac{\Delta}{2}\right\}$$

$$Q_3=\left\{(\Delta,x)\,\middle|\,x_*+\frac{\Delta}{2}<x\right\}.$$

$$I_A=\begin{cases}1 & , \ if\ (\Delta,x)\in A\\ 0 & , \ if\ (\Delta,x)\notin A\end{cases}.$$

Then, property 2.6 can be written as following.

Property 2.7 For the condition $x-\Delta<x_*<x$ and $0<\Delta\le\frac{a}{-b}$, the centroid of (52)~(54) of $\mu_{N(\tilde{D})}(z)$ is

$$M_4(\Delta,x)=\sum_{k=1}^{3}M_{3k}(\Delta,x)\cdot I_{Q_k} \tag{58}$$

2.4 $\mu_{N(\tilde{a})}(z)$ AND ITS CENTROID UNDER THE CONDITION $0<x_1<x_2<x_*<x_3$

In condition, $0<x_1<x_2<x_*<x_3$ (59)

from (14), we obtain

$$e_1<e_2. \tag{60}$$

As §2.3, we obtain the following.

All arrangements for e_1, e_2, and e_3, under condition (60), we obtain from (14) the following relationship:

$$e_1<e_2<e_3 \text{ if } 0<x<x_*-\frac{\Delta}{2} \ (\Leftrightarrow \text{since } x_2+x_3<2x_*) \tag{61}$$

$e_1 < e_3 < e_2$ if $x_* - \frac{\Delta}{2} < x < x_* (\Leftrightarrow \text{since } x_2 + x_3 > 2x_* \text{ and}$ (62)
$x_1 + x_2 < 2x_*)$

$e_3 < e_1 < e_2$ if $x_* < x (\Leftrightarrow \text{since } x_1 + x_3 > 2x_*)$ (63)

From table 1, (13)~(20) and (59)~(63), we obtain the following:

(1)case (1), (4), (10), $\mu_{N(\tilde{D})}(z) = 0$.

(2)case (2), (5) there is an inequality $d_2 \le x_2$ and $x_2 < x_*$, from (4°)(20) no solution.

(3)case(3), $d_1 \le x_1$ and $x_2 < d_2 < x_3$, since $x_j < x_* < x_3$, $j = 1,2$; therefore, from (16), (18), (20), we obtain $z \le e_1$, $z \le -u + \frac{(a-v)^2}{4(-b)}$ and $e_3 \le z \le -u + \frac{(a-v)^2}{4(-b)}$, from (63) and (13),

when $x_* < x$, then $\mu_{N(\tilde{D})}(z) = \frac{x + \Delta - d_2}{\Delta}$ if $e_3 \le z \le e_1$ (64)

(4)case(6) $x_1 \le d_1 \le x_2 \le d_2 \le x_3$, from (15), (16), (18), (20), then

$e_1 \le z \le -u + \frac{(a-v)^2}{4(-b)}$, $z \le e_2$, $z \le -u + \frac{(a-v)^2}{4(-b)}$ and $e_3 \le z \le -u + \frac{(a-v)^2}{4(-b)}$,

from (62), (63) and (12), (13), we obtain

when $x_* - \frac{\Delta}{2} < x < x_*$, then

$$\mu_{N(\tilde{D})}(z) = \max\left[\frac{d_1 - x + \Delta}{\Delta}, \frac{x + \Delta - d_2}{\Delta}\right] = \frac{d_1 - x + \Delta}{\Delta} \text{ if } e_3 \le z \le e_2 \quad (65)$$

when $x_* < x$, then

$$\mu_{N(\tilde{D})}(z) = \max\left[\frac{d_1 - x + \Delta}{\Delta}, \frac{x + \Delta - d_2}{\Delta}\right] = \frac{x + \Delta - d_2}{\Delta} \text{ if } e_1 \le z \le e_2 \quad (66)$$

(5)case(7) $x_1 \le d_1 \le x_2$ and $x_3 \le d_2$, from (15), (16), (19), we obtain

$e_1 \le z \le -u + \frac{(a-v)^2}{4(-b)}$, $z \le e_2$ and $z \le e_3$, from (61), (62) and (13), we obtain

when $0 < x < x_* - \frac{\Delta}{2}$, $\mu_{N(\tilde{D})}(z) = \frac{d_1 - x + \Delta}{\Delta}$ if $e_1 \le z \le e_2$ (67)

when $x_* - \frac{\Delta}{2} < x < x_*$, then $\mu_{N(\tilde{D})}(z) = \frac{d_1 - x + \Delta}{\Delta}$ if $e_1 \le z \le e_3$ (68)

(6)case(8) $x_2 \le d_1$ and $d_2 \le x_3$, from (15), (20), we obtain $e_2 \le z \le -u + \frac{(a-v)^2}{4(-b)}$

and $e_3 \le z \le -u + \frac{(a-v)^2}{4(-b)}$; from (61)~(63) and (13), we obtain

when $0 < x < x_* - \frac{\Delta}{2}$, then $\mu_{N(\tilde{D})}(z) = \frac{x + \Delta - d_1}{\Delta}$ if $e_3 \le z \le -u + \frac{(a-v)^2}{4(-b)}$ (69)

when $x_* - \frac{\Delta}{2} < x < x_*$, then $\mu_{N(\tilde{D})}(z) = \frac{x + \Delta - d_1}{\Delta}$ if $e_2 \le z \le -u + \frac{(a-v)^2}{4(-b)}$ (70)

when $x_* < x$, then $\mu_{N(\tilde{D})}(z) = \dfrac{x+\Delta-d_1}{\Delta}$ if $e_2 \le z \le -u + \dfrac{(a-v)^2}{4(-b)}$ (71)

(7)case(9) $x_2 \le d_1 \le x_3 \le d_2$, from (15), (17), (19), we obtain $e_2 \le z \le -u + \dfrac{(a-v)^2}{4(-b)}$,

$z \le -u + \dfrac{(a-v)^2}{4(-b)}$ and $z \le e_3$;

from (61) and (13), we obtain

when $0 < x < x_* - \dfrac{\Delta}{2}$, then $\mu_{N(\tilde{D})}(z) = \dfrac{x+\Delta-d_1}{\Delta}$ if $e_2 \le z \le e_3$ (72)

Summarizing above, we obtain the following property 2.8.

When $0 < \Delta < x \le \dfrac{a}{-b}$, $0 < x_1 < x_2 < x_3$ is valid, $x_1 = x - \Delta$, $x_2 = x$, $x_3 = x + \Delta$, therefore $0 < x_1 < x_2 < x_* < x_3$ can be converted into a condition $x < x_* < x + \Delta$ and $0 < \Delta < x \le \dfrac{a}{-b}$.

Property 2.8 For the condition $x < x_* < x + \Delta$ and $0 < \Delta < x \le \dfrac{a}{-b}$, the membership function of $N(\tilde{D})$ is

(1°)if $0 < x < x_* - \dfrac{\Delta}{2}$, $(e_1 < e_2 < e_3)$ then, from (67), (72), (69), we obtain

$$\mu_{N(\tilde{D})}(z) = \begin{cases} \dfrac{d_1 - x + \Delta}{\Delta} & , \quad e_1 \le z \le e_2 \\ \dfrac{x+\Delta-d_1}{\Delta} & , \quad e_2 \le z \le -u + \dfrac{(a-v)^2}{4(-b)} \\ 0 & , \quad \textit{otherwise} \end{cases} \tag{73}$$

(2°)if $x_* - \dfrac{\Delta}{2} < x < x_*$, $(e_1 < e_3 < e_2)$ then, from (68), (65), (70), we obtain

$$\mu_{N(\tilde{D})}(z) = \begin{cases} \dfrac{d_1 - x + \Delta}{\Delta} & , \quad e_1 \le z \le e_2 \\ \dfrac{x+\Delta-d_1}{\Delta} & , \quad e_2 \le z \le -u + \dfrac{(a-v)^2}{4(-b)} \\ 0 & , \quad \textit{otherwise} \end{cases} \tag{74}$$

(3°)if $x_* < x$, $(e_3 < e_1 < e_2)$ then, from (64), (66), (71), we obtain

$$\mu_{N(\tilde{D})}(z) = \begin{cases} \dfrac{x+\Delta-d_2}{\Delta} & , \quad e_3 \le z \le e_2 \\ \dfrac{x+\Delta-d_1}{\Delta} & , \quad e_2 \le z \le -u + \dfrac{(a-v)^2}{4(-b)} \\ 0 & , \quad \textit{otherwise} \end{cases} \tag{75}$$

Property 2.9 For the condition $x < x_* < x + \Delta$ and $0 < \Delta < x \le \frac{a}{-b}$, the centroid of (73)~(75) of $\mu_{N(\tilde{D})}(z)$ is

(4°) if $0 < x < x_* - \frac{\Delta}{2}$, the centroid of (73) of $\mu_{N(\tilde{D})}(z)$ is

$$M_{41}(\Delta, x) = \frac{R_{41}(\Delta, x)}{P_{41}(\Delta, x)}. \tag{76}$$

At here,

$$P_{41}(\Delta, x) = \int_{-\infty}^{\infty} \mu_{N(\tilde{D})}(z)dz$$
$$= \frac{1}{\Delta}V_1(e_1, e_2) + \frac{-x+\Delta}{\Delta}(e_2 - e_1) + \frac{x+\Delta}{\Delta}\left(-u + \tfrac{(a-v)^2}{4(-b)} - e_2\right) - \frac{1}{\Delta}V_1\left(e_2, -u + \tfrac{(a-v)^2}{4(-b)}\right)$$

$$R_{41}(\Delta, x) = \int_{-\infty}^{\infty} z \cdot \mu_{N(\tilde{D})}(z)dz$$
$$= \frac{1}{\Delta}V_{12}(e_1, e_2) + \frac{-x+\Delta}{2\Delta}\left(e_2^2 - e_1^2\right) + \frac{x+\Delta}{2\Delta}\left[\left(-u + \tfrac{(a-v)^2}{4(-b)}\right)^2 - e_2^2\right] - \frac{1}{\Delta}V_{12}\left(e_2, -u + \tfrac{(a-v)^2}{4(-b)}\right)$$

(5°) if $x_* - \frac{\Delta}{2} < x < x_*$, the centroid of (74) of $\mu_{N(\tilde{D})}(z)$ is

$$M_{42}(\Delta, x) = \frac{R_{42}(\Delta, x)}{P_{42}(\Delta, x)}. \tag{77}$$

At here,

$$P_{42}(\Delta, x) = \int_{-\infty}^{\infty} \mu_{N(\tilde{D})}(z)dz$$
$$= \frac{1}{\Delta}V_1(e_1, e_2) + \frac{-x+\Delta}{\Delta}(e_2 - e_1) + \frac{x+\Delta}{\Delta}\left(-u + \tfrac{(a-v)^2}{4(-b)} - e_2\right) - \frac{1}{\Delta}V_1\left(e_2, -u + \tfrac{(a-v)^2}{4(-b)}\right)$$

$$R_{42}(\Delta, x) = \int_{-\infty}^{\infty} z \cdot \mu_{N(\tilde{D})}(z)dz$$
$$= \frac{1}{\Delta}V_{12}(e_1, e_2) + \frac{-x+\Delta}{2\Delta}\left(e_2^2 - e_1^2\right) + \frac{x+\Delta}{2\Delta}\left[\left(-u + \tfrac{(a-v)^2}{4(-b)}\right)^2 - e_2^2\right] - \frac{1}{\Delta}V_{12}\left(e_2, -u + \tfrac{(a-v)^2}{4(-b)}\right)$$

(6°) if $x_* < x$, the centroid of (75) of $\mu_{N(\tilde{D})}(z)$ is

$$M_{43}(\Delta, x) = \frac{R_{33}(\Delta, x)}{P_{33}(\Delta, x)}. \tag{78}$$

At here,

$$P_{43}(\Delta, x) = \int_{-\infty}^{\infty} \mu_{N(\tilde{D})}(z)dz$$
$$= \frac{x+\Delta}{\Delta}(e_2 - e_3) - \frac{1}{\Delta}V_2(e_3, e_2) + \frac{x+\Delta}{\Delta}\left[\left(-u + \tfrac{(a-v)^2}{4(-b)}\right)^2 - e_2\right] - \frac{1}{\Delta}V_1\left(e_2, -u + \tfrac{(a-v)^2}{4(-b)}\right)$$

$$R_{43}(\Delta, x) = \int_{-\infty}^{\infty} z \cdot \mu_{N(\tilde{D})}(z)dz$$
$$= \frac{x+\Delta}{2\Delta}\left(e_2^2 - e_3^2\right) - \frac{1}{\Delta}V_{22}(e_3, e_2) + \frac{x+\Delta}{2\Delta}\left[\left(-u + \tfrac{(a-v)^2}{4(-b)}\right)^2 - e_2^2\right] - \frac{1}{\Delta}V_{12}\left(e_2, -u + \tfrac{(a-v)^2}{4(-b)}\right).$$

Assume,

$$E_1 = \left\{(\Delta, x) \middle| 0 < x < x_* - \frac{\Delta}{2}\right\}$$

$$E_2 = \left\{(\Delta, x) \middle| x_* - \frac{\Delta}{2} < x < x_*\right\}$$

$$E_3 = \{(\Delta, x) | x_* < x\}$$

then, property 2.9 can be written as following.

Property 2.10 For the condition $x < x_* < x + \Delta$ and $0 < \Delta \le \frac{a}{-b}$, the centroid of (73)~(75) of $\mu_{N(\tilde{D})}(z)$ is

$$M_4(\Delta, x) = \sum_{k=1}^{3} M_{4k}(\Delta, x) \cdot I_{E_k}. \tag{79}$$

3. OPTIMAL SOLUTION

From property 2.2 (30), property 2.4 (36), property 2.7 (58), property 2.10 (79) and assume the following, then we obtain the theorem 1.

Assume,

$$R_1 = \left\{(\Delta, x) \middle| x_* < x - \Delta \text{ and } 0 < \Delta < x \le \frac{a}{-b}\right\}$$

$$R_2 = \left\{(\Delta, x) \middle| x + \Delta < x_* \text{ and } 0 < \Delta < x \le \frac{a}{-b}\right\}$$

$$R_3 = \left\{(\Delta, x) \middle| x - \Delta < x_* < x \text{ and } 0 < \Delta < x \le \frac{a}{-b}\right\}$$

$$R_4 = \left\{(\Delta, x) \middle| x < x_* < x + \Delta \text{ and } 0 < \Delta < x \le \frac{a}{-b}\right\}.$$

Theorem 1 The Centroid of $\mu_{N(\tilde{D})}(z)$ is

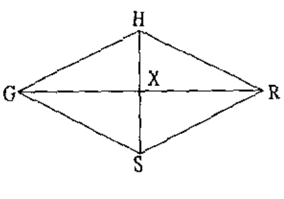
$$M(\Delta, x) = \sum_{k=1}^{3} M_k(\Delta, x) \cdot I_{R_k} \tag{80}$$

Under the condition $0 < \Delta < x \le \frac{a}{-b}$ the estimated profit value of demand quantity x in the fuzzy sense is $M(\Delta, x)$.

In order to minimize $-M(\Delta, x)$ [i.e. maximize $M(\Delta, x)$], we apply the Nelder-Mead method[3]. But in our paper the Δ, x should satisfy $0 < \Delta < x \le \frac{a}{-b}$, therefore, we apply the Nelder-Mead simplex algorithm [1], the two transformation (81) and (82) shown in figures 1 and 2, instead of the two transformation of algorithm 6.5 of Nelder-Mead method[3].

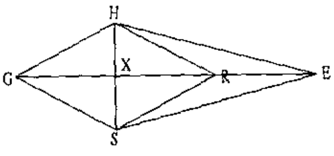
In this paper, we denote Δ for $R(1)$, $X(1)$, $G(1)$, $E(1)$ and x for $R(2)$, $X(2)$, $G(2)$, $E(2)$.

optimal point for the last time



$$R = X + d(X - G) = (1 + d)X - dG \quad \text{where } 0 < d \le 1 \tag{81}$$

optimal point for the last time



$$E = X + e(R - X) = (1-e)X + eK \quad \text{where } e > 1 \tag{82}$$

Fig 2. Contraction step

We let $H = X(1) - X(2) - G(1) + G(2)$

$$I(H) = \begin{cases} 1 & if \quad H > 0 \\ 0 & if \quad H \le 0 \end{cases}$$

$$H_* = \min\left[\frac{X(2) - X(1)}{H} \cdot I(H), 1\right].$$

If we take in (81) satisfying

$$0 < d < H_* \tag{83}$$

then it is easy to shown that $R(1) < R(2)$.

We let $L = X(2) - X(1) + R(1) - R(2)$ and

$$L_* = \begin{cases} \dfrac{X(2) - X(1)}{L} & if \quad L > 0 \\ \infty & if \quad L \le 0 \end{cases}$$

If we take d in (83) satisfying

$$1 < e < L_* \tag{84}$$

then it is easy to shown that $E(1) < E(2)$.

We modify $R(K) = 2M(k) - V(Hi, K)$ in the subroutine Newpoints of Algorithm 6.5[3] to be $R(K) = (1+d)M(K) - dV(Hi, K)$ where d satisfies (83). Also, we modify $E(K) = 2R(K) - M(K)$ to be $E(K) = e \cdot R(K) + (1-e) \cdot M(K)$, where e satisfies (84). Use the modified Algoritme 6.5[3], we can find Δ^{**}, x^{**} such that -$M(\Delta^{**}, x^{**})$ is the local minimal value that is $M(\Delta^{**}, x^{**})$ is the local maximal value. The $M_0 = x^{**}$ is optimal demand and $M(\Delta^{**}, x^{**})$ is the maximum profit in fuzzy sense.

4. NUMERICAL EXAMPLE IMPLEMENTATION

In this section, we apply theorem 3 for some numerical examples to find optimal demand x^{**} in the fuzzy sense such that estimated profit $M(\Delta^{**}, x^{**})$ is the maximum.

Let (Δ, x) be any initial points, Δ^{**}, x^{**} the coordinates of local minimum for -$M(\Delta, x)$, x^{**} the centroid (the optimal demand) for the triangular fuzzy number $(x^{**} - \Delta^{**}, x^{**}, x^{**} + \Delta^{**})$ $x_* = \dfrac{a - v}{2(-b)}$ is the crisp optimal demand and profit

$N(x_*) = -u + \dfrac{(a-v)^2}{4(-b)}$ is the maximum profit. Let $r_D = \dfrac{x^{**} - x_*}{x_*} \times 100\%$,

$r_N = \dfrac{M(\Delta^{**}, x^{**}) - N(x_*)}{N(x_*)} \times 100\%$ be the relative error of demand in the fuzzy sense and the relative error of fuzzy profit.

From the definition $\mu_{N(\tilde{D})}(x^{**})$, we know that it is the membership degree of x^{**}.

EXAMPLE 4.1 Since there are two variable in $M(\Delta, x)$ by algorithm discussed in section 3. When we run the program to solve assign a set of three initial points (Δ, x) which satisfies $0 < \Delta < x \le \frac{a}{-b}$. Given $a = 30$, $b = -2$, $u = 10$, $v = 6$, $x_* = 6$, $N(x_*) = 62$, we give set (4.1.1)~(4.1.5) of initial points value of (Δ, x) and we obtain the computing results for each case as follows.

From §2(9), the range of $\mu_{N(\tilde{D})}(z)$ is $z \le N(x_*)$; therefore, it is necessary to find (Δ, x) to make $M(\Delta, x)$ the closer to $N(x_*)$ the better.

(4.1.1)

Δ	X	
0.8715	5.2360	$\Delta^{**} = 0.0656$
0.4412	6.5568	$X^{**} = 5.9959$
1.6567	5.4938	$M(\Delta^{**}, X^{**}) = 61.997084163$
		$r_d = 0.068333\%$
		$r_N = 0.004703\%$

(4.1.2)

Δ	X	
0.9934	6.1403	$\Delta^{**} = 0.0518$
0.9083	5.1739	$X^{**} = 5.9912$
1.5070	5.1229	$M(\Delta^{**}, X^{**}) = 61.997782973$
		$r_d = 0.146667\%$
		$r_N = 0.003576\%$

(4.1.3)

Δ	X	
0.8643	5.3484	$\Delta^{**} = 0.003$
0.3056	6.4804	$X^{**} = 6.0005$
1.5147	5.5550	$M(\Delta^{**}, X^{**}) = 61.99959915$
		$r_d = 0.008333\%$
		$r_N = 0.000065\%$

(4.1.4)

Δ	X	
1.5216	5.6120	$\Delta^{**} = 0.0038$
0.8353	5.8086	$X^{**} = 6.0083$
0.1538	6.9196	$M(\Delta^{**}, X^{**}) = 61.99844747$
		$r_d = 0.138333\%$
		$r_N = 0.000252\%$

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