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模糊化的更新報酬過程及其應用

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中文摘要

關鍵詞：模糊集、模糊隨機變數、模糊更新過程、模糊更新報酬過程

考慮更新報酬過程中報酬及區間到達時間爲模糊性時證得在長期平均模糊報酬，正爲單一週期的模糊期望報酬除以單一週期的模糊期望週長

英文摘要

Keyword: Fuzzy set; fuzzy random variable; the strong law of large numbers; a fuzzy renewal process; a renewal reward process.

In this paper, we consider a renewal reward processes with fuzzy reward and inter-arrival time in the fuzzy sense by using a metric and prove a proposition which the long run average fuzzy reward with inter-arrival time in the fuzzy sense is just the expected reward earned during a cycle divided by the expected length of the cycle in the fuzzy sense.

1.Introduction

The notion of a fuzzy set was introduced by Zadeh [11] who defined it as a generalized characteristic function; that is, one which varies between zero and one rather than merely assuming the two values of zero and one. Intermediate values give grades of membership of various points in the set, higher values implying higher grades of membership. Typical examples of fuzzy sets are the set of numbers around 5 or the set of the times of arrival. The concept of fuzzy random variables provided by Kwakernaak [3,4] , Puri and Ralescc[5]is a particular kind of fuzzy set. An SLLN for sums of independent and identical distributed fuzzy random variables was obtained by Kruse [7] and Kim[10] proved an SLLN for sums of independent and identical distributed fuzzy random variables by using a metric. The concept of a renewal process,having inter-arrival time in the fuzzy sense, was provided by Hwang[1]. He also showed a result for the rate of a renewal process in the fuzzy sense. Renewal reward processes with fuzzy rewards were provided by Popova and Wu[2]. In this paper, we consider renewal reward processes with fuzzy rewards and inter-arrival times in the fuzzy sense by using a metric and prove a proposition which the long run average fuzzy reward with inter-arrival times in the fuzzy sense is just the expected reward earned during a cycle divided by the expected length of the cycle in the fuzzy sense.

Section 2 describes briefly some properties of fuzzy numbers and defines fuzzy random variables. the expectation of fuzzy random variables and a metric d on the set of fuzzy numbers F(R). In section 3,a renewal process having inter-arrival times which are fuzzy random variables and a theorem for the rate of a renewal process having inter-arrival times which are fuzzy random variables are reference from Hwang[1]. In section 4, we consider a renewal reward process with fuzzy rewards and inter-arrival times in the fuzzy sense by using a metric and prove a proposition which shows that the long run average fuzzy reward with inter-arrival times in the fuzzy sense is just the expected reward earned during a cycle divided by the expected length of the cycle in the fuzzy sense. Finally an example is provided.

2. Fuzzy numbers, fuzzy random variables and the expectation of fuzzy random variables.

If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{M}$ in X is a set of ordered pairs: $\tilde{M} = \{(x,\tilde{M}(x))|x \in X\}$,where $\tilde{M}(x)$ is called the membership function or grade of membership. We have the following definition by Zimmermann [12].

Definition 2.1

A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line R such that

a. It exists exactly one $x_0 \in R$ with the membership function $\tilde{M}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$).
b. The membership function $\tilde{M}(x)$ is piecewise continuous.

Definition 2.2

The crisp set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α -level-set : $\tilde{A}_\alpha = \{x : \tilde{A}(x) \geq \alpha\}, 0 \leq \alpha \leq 1.$ (2.1)

The α -level-set $\tilde{A}_\alpha$ is a useful tool for the treatment of fuzzy numbers.

Definition 2.3

If the α -level-set $\tilde{A}_\alpha$ is a closed and bounded set for all $\alpha \in (0,1]$, then the fuzzy set $\tilde{A}$ is said to be closed and bounded set.

A fuzzy number $\tilde{M}$ may be decomposed into its α -level-sets by

$$\tilde{M} = \sup_{\alpha \in [0,1] \cap Q} \{\alpha \cdot I_{\tilde{M}_\alpha}\} \quad (2.2)$$

or

$$\tilde{M}(x) = \sup_{\alpha \in (0,1] \cap Q} \{\alpha \cdot I_{\tilde{M}_\alpha}(x)\} \quad (2.3)$$

where Q is the set of the rational number.

Let $\tilde{U}$ and $\tilde{V}$ be two fuzzy numbers in R. By the extension principle, four operations on fuzzy numbers $\tilde{U}$ and $\tilde{V}$ are defined as follows:

$$(\tilde{U} + \tilde{V})(t) = \sup_{x+y=t} \{\tilde{U}(x) \wedge \tilde{V}(y)\} \quad (2.4)$$

$$(\tilde{U} - \tilde{V})(t) = \sup_{x-y=t} \{\tilde{U}(x) \wedge \tilde{V}(y)\} \quad (2.5)$$

$$(\tilde{U} * \tilde{V})(t) = \sup_{x \cdot y=t} \{\tilde{U}(x) \wedge \tilde{V}(y)\} \quad (2.6)$$

$$\left(\frac{\tilde{U}}{\tilde{V}}\right)(t) = \sup_{x=t \cdot y} \{\tilde{U}(x) \wedge \tilde{V}(y)\} \quad (2.7)$$

By R.Kruse and K.D.Meyer [8],we have the following Lemma2.1.

Lemma 2.1.

Let $n \in N$ and $f : R^n \to R$ be a mapping. For any subsets $A_1, A_2, ..., A_n \subseteq R$ define
$f(A_1, A_2, ..., A_n) = \{t \in R \mid \exists (t_1, t_2, ..., t_n) \in (A_1, A_2, ..., A_n) \text{ with } f(t_1, t_2, ..., t_n) = t\}$
Assume $\tilde{U}_i$ is a closed and bounded fuzzy random variable, i=1,…n,then
$(f(\tilde{U}_1, \tilde{U}_2, ..., \tilde{U}_n))_\alpha = f((\tilde{U}_1)_\alpha, (\tilde{U}_2)_\alpha, ..., (\tilde{U}_n)_\alpha)$

Let $X = [x^-, x^+] = \{x | x^- \leq x \leq x^+\}$ and $Y = [y^-, y^+] = \{y | y^- \leq y \leq y^+\}$

for $x^-, x^+, y^-, y^+ \in R$.

Four operations (+, −, $*$ and /) on X and Y are defined as follows:

$$X + Y = [x^-, x^+] + [y^-, y^+] = [x^- + y^-, x^+ + y^+] \tag{2.8}$$

$$X - Y = [x^-, x^+] - [y^-, y^+] = [x^- - y^+, x^+ - y^-] \tag{2.9}$$

$$\begin{aligned} X * Y &= [x^-, x^+] * [y^-, y^+] \\ &= [\min(x^- y^-, x^+ y^+, x^- y^+, x^+ y^-), \max(x^- y^-, x^+ y^+, x^- y^+, x^+ y^-)] \end{aligned} \tag{2.10}$$

and $$\frac{X}{Y} = [x^-, x^+] * [\frac{1}{y^+}, \frac{1}{y^-}], 0 < y^- < y^+ \tag{2.11}$$

Let F(R) be the set of fuzzy numbers in R.

Theorem 2.1 (Goetsdhel and Voxman[6])

For $\tilde{u} \in F(R)$, denote $\tilde{u}^L(\alpha) = \tilde{u}^L_\alpha$ and $\tilde{u}^U(\alpha) = \tilde{u}^U_\alpha$ by considering as functions of $\alpha \in [0,1]$. Then

(1) $\tilde{u}^L(\alpha)$ is a bounded increasing function on [0,1].

(2) $\tilde{u}^U(\alpha)$ is a bounded decreasing function on [0,1].

(3) $\tilde{u}^L(1) \le \tilde{u}^U(1)$.

(4) $\tilde{u}^L(\alpha)$ and $\tilde{u}^U(\alpha)$ are left continuous on (0,1) and right continuous at 0.

(5) If $\tilde{v}^L(\alpha)$ and $\tilde{v}^U(\alpha)$ satisfy above (1)-(4), then there exists a unique $\tilde{v} \in F(R)$ such that $\tilde{v}_\alpha = [\tilde{v}^L(\alpha), \tilde{v}^U(\alpha)]$.

Definition 2.4

Let (Ω, Γ, P) be a probability space and let F(R) be the set of fuzzy numbers in R (i.e. for ω $\Omega, \tilde{X}(\omega) \in F(R)$) . A fuzzy random variable is a function $\tilde{X}: \Omega \to F(R)$ such that $\{(\omega, x) : x \in \tilde{X}(\omega)_\alpha (x) \ge \alpha\} \in \Gamma \times ß$(where ß denotes the Borel subsets of R) for every $\alpha \in [0,1]$, where $\tilde{X}(\omega)_\alpha = \{x \in R : \tilde{X}(\omega)(x) \ge \alpha$.

Let (Ω, Γ, P) be a probability space and let $\tilde{X}_t : \Omega \to F(R)$ be a fuzzy random variable, for each $t \in T$, an index set. We called $\{\tilde{X}_t\}_{t \in T}$ a fuzzy stochastic process.

Let $\tilde{X}_i$, i= 1,2,... be a sequence of fuzzy random variables on (Ω ,Γ , P).

If for $\alpha \in (0,1]$, $\omega \in \Omega$, $\tilde{X}^L_{i,\alpha}$ and $\tilde{X}^U_{i,\alpha}$ defined by

$$\tilde{X}^L_{i,\alpha}(\omega) = \inf \tilde{X}_i(\omega)_\alpha$$

$$\text{and } \tilde{X}^U_{i,\alpha}(\omega) = \sup \tilde{X}_i(\omega)_\alpha$$

are sequences of independent and identical distributed crisp random variables, then the $\tilde{X}_i$, i= 1,2,... is called a sequence of independent and identical distributed fuzzy random variables.

Definition 2.5

Let (Ω, Γ, P) be a probability space and let $\widetilde{X}:\Omega \to F(R)$ be a fuzzy random variable. If for each $\alpha \in [0,1]$, $\widetilde{X}_\alpha^L$ and $\widetilde{X}_\alpha^U$ are integrable ,the expectation of a fuzzy random variable $\widetilde{X}$ is the fuzzy number $E\widetilde{X}$ such that

$$\{x \in R : E\widetilde{X}(x) \geq \alpha\} = [E\widetilde{X}_\alpha^L, E\widetilde{X}_\alpha^U]$$

By Definition 2.5 and (2.3), we get $E\widetilde{X}(x) = \sup_{0 \leq \alpha \leq 1} \alpha \, I_{[E\widetilde{X}_\alpha^L, E\widetilde{X}_\alpha^U]}(x)$ (2.12)

Let $\widetilde{X}_1$ and $\widetilde{X}_2$ be fuzzy random variables on (Ω, Γ, P),we define the sum $\widetilde{X}_1 + \widetilde{X}_2$ and the product $\widetilde{X}_1 * \widetilde{X}_2$ as follows:

$$(\widetilde{X}_1 + \widetilde{X}_2)(\omega) = \widetilde{X}_1(\omega) + \widetilde{X}_2(\omega) \tag{2.13}$$

$$(\widetilde{X}_1 * \widetilde{X}_2)(\omega) = \widetilde{X}_1(\omega) * \widetilde{X}_2(\omega) \tag{2.14}$$

Let F(R) be the set of fuzzy numbers and $\widetilde{u} \in F(R)$.
We define:

$$\widetilde{u}_\alpha^L = \inf \widetilde{u}_\alpha$$

and $$\widetilde{u}_\alpha^U = \sup \widetilde{u}_\alpha \, .$$

From this characterization of fuzzy numbers, α-level-set $\widetilde{u}_\alpha = [\widetilde{u}_\alpha^L, \widetilde{u}_\alpha^U]$.
A metric d,on F(R) is defined by

$d(\widetilde{u}, \widetilde{v}) = \sup_{0 \leq \alpha \leq 1} d_H(\widetilde{u}_\alpha, \widetilde{v}_\alpha)$,where d_H is the Hausdorff metric defined as

$$d_H(\widetilde{u}_\alpha, \widetilde{v}_\alpha) = \max(|\widetilde{u}_\alpha^L - \widetilde{v}_\alpha^L|, |\widetilde{u}_\alpha^U - \widetilde{v}_\alpha^U|).$$

Also,the norm $\|\widetilde{u}\|$ of fuzzy number $\widetilde{u}$ will be defined as

$$\|\widetilde{u}\| = d(\widetilde{u}, \widetilde{0}) = \max(|\widetilde{u}_0^L|, |\widetilde{u}_0^U|).$$

3. The renewal process

Let (Ω, Γ, P) be a probability space.In crisp renewal process, let the random variable X_n be the time between the (n-1)st and the nth event of a process such that $X_n : \Omega \to R$, $n \geq 1$. If random variables $\{X_1, X_2, \dots$ are independent and identical distributed, then $S_n = \sum_{i=1}^{n} X_i$ represents the occurred time of nth event. For example a radio works on a single battery.As soon as the battery in use fails, it is immediately renewed with a new one. Suppose that we have an infinite supply of batteries whose lifetimes are independent and identical distributed random variables $X_1, X_2, \dots$,where $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$, then $S_1 = X_1$ is the time of the first renewal; $S_2 = X_1 + X_2$ is the time of the second renewal. In general, S_n denotes the time of the nth renewal. In this crisp sense, the lifetimes of a battery are considered by crisp languages. But the lifetimes of a battery are

uncertainty in real situations. Therefore, we would rather consider the lifetime of a battery in fuzzy languages than crisp languages.
For a probability space (Ω , Γ, P), let the time between the (n-1)st and the nth event of a process in the fuzzy sense be a fuzzy random variable $\widetilde{X}_n$ such that $\widetilde{X}_n : \Omega \rightarrow F(R^+)$, where $F(R^+)$ is the set of fuzzy numbers in R^+ (i.e. for $\omega \in \Omega$, $\widetilde{X}_n(\omega)$ $F(R^+)$ and $R^+=(0,\infty)$) and $n \geq 1$. For each $\alpha \in (0,1)$, both $\widetilde{X}^L_{i,\alpha}$ and $\widetilde{X}^U_{i,\alpha}$ defined by

$$\widetilde{X}_{i}{}^L_\alpha(\omega) = \inf\left\{t|\widetilde{X}_i(\omega)(t) \geq \alpha\right\}$$

and $\widetilde{X}_{i}{}^U_\alpha(\omega) = \sup\left\{t|\widetilde{X}_i(\omega)(t) \geq \alpha\right\}$, for i= 1,2,... are sequences of independent and identical distributed random variables.
We have the α -level-set $\widetilde{X}_i(\omega)_\alpha = [\widetilde{X}_{i}{}^L_\alpha(\omega), \widetilde{X}_{i}{}^U_\alpha(\omega)]$, i= 1,2,...
By the strong law of large numbers in the crisp sense, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \widetilde{X}^L_{i,\alpha} = E\widetilde{X}^L_{1,\alpha} \quad \text{a.s.}$$

$$\text{and} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \widetilde{X}^U_{i,\alpha} = E\widetilde{X}^U_{1,\alpha} \quad \text{a.s.}$$

For a sequence of independent and identical distributed fuzzy random variables $\{\widetilde{X}_1, \widetilde{X}_2, \ldots$, let $\widetilde{S}_n = \sum_{i=1}^{n} \widetilde{X}_i$, that is , $\widetilde{S}_n$ represents the occurred time of nth renewal in the fuzzy sense. For each $\alpha \in (0,1]$, each $\omega \in \Omega$ and $n \in N$, $\widetilde{S}_n(\omega)^L_\alpha$ and $\widetilde{S}_n(\omega)^U_\alpha$ are defined by

$$\widetilde{S}_n(\omega)^L_\alpha = \inf\left\{t|\widetilde{S}_n(\omega)(t) \geq \alpha\right\} \tag{3.1}$$

$$\text{and} \quad \widetilde{S}_n(\omega)^U_\alpha = \sup\left\{t|\widetilde{S}_n(\omega)(t) \geq \alpha\right\}. \tag{3.2}$$

Hence, the α -level-set $(\widetilde{S}_n(\omega))_\alpha = [\widetilde{S}_n(\omega)^L_\alpha, \widetilde{S}_n(\omega)^U_\alpha]$.
We also let $\widetilde{N}^U_\alpha(t)(\omega) = \sup\{n \mid \widetilde{S}_n(\omega)^L_\alpha \leq t$ (3.3)
and $\widetilde{N}^L_\alpha(t)(\omega) = \sup\{n \mid \widetilde{S}_n(\omega)^U_\alpha \leq t$. (3.4)
It is clear that $\widetilde{N}^U_\alpha(t) \geq \widetilde{N}^L_\alpha(t)$.

Definition3.1

Let the time between the (n-1)st and the nth event of a process in the fuzzy sense be a fuzzy random variable $\widetilde{X}_n$, $n \geq 1$. If $\{\widetilde{X}_1, \widetilde{X}_2, \ldots$ is a sequence of independent and identical distributed fuzzy random variables, then the fuzzy stochastic process $\left\{\widetilde{N}(t), t \geq 0\right\}$ is said to be a fuzzy renewal process where $\widetilde{N}(t)$ is a fuzzy random variable with the following fuzzy number defined by, for each $\omega \in \Omega$,

$$\widetilde{N}(t)(\omega) = \sup_{\alpha \in (0,1] \cap Q} \left\{\alpha \cdot I_{[\widetilde{N}^L_\alpha(t)(\omega), \widetilde{N}^U_\alpha(t)(\omega)]}\right.$$, Q is the set of the rational numbers.

It means that the fuzzy random variable $\widetilde{N}(t)$ represents the total number of "events" that have renewal to time t in the fuzzy sense. The following theorem was provided by Hwang[1]

Theorem 3.1

With probability one, $\dfrac{\tilde{N}(t)}{t} \rightarrow \dfrac{1}{E\tilde{X}_1}$ as $t \rightarrow \infty$.

4. **The renewal reward process with inter-arrival times and a reward in the fuzzy sense**

Let the time between the (n-1)st and the nth event of a process in the fuzzy sense be a fuzzy random variable $\tilde{X}_n$, $n \geq 1$ and let a fuzzy renewal process be defined by $\{\tilde{N}(t), t \geq 0\}$ where $\tilde{N}(t) = \sup_{\alpha \in (0,1] \cap Q} \{\alpha \cdot I_{[\tilde{N}^L_\alpha(t), \tilde{N}^U_\alpha(t)]}$, Q is the set of the rational number.

Suppose that each time a renewal occurs, we receive a reward. We denote by $\tilde{R}_n$ the fuzzy reward earned at the time of the nth renewal. We shall assume that $\{\tilde{R}_n, n \geq 1$ is a sequence of independent and identical distributed fuzzy random variables.

For each $\alpha \in (0,1)$, both $\tilde{R}^L_{n\alpha}$ and $\tilde{R}^U_{n\alpha}$ are defined by:

For each $\omega \in \Omega$,

$$\tilde{R}^L_{n,\alpha}(\omega) = \inf\{s \mid \tilde{R}_n(\omega)(s) \geq \alpha\}$$

and $$\tilde{R}^U_{n,\alpha}(\omega) = \sup\{s \mid \tilde{R}_n(\omega)(s) \geq \alpha\}$$

Hence, the α-level-set $(\tilde{R}_n(\omega))_\alpha = [\tilde{R}^L_{n,\alpha}(\omega), \tilde{R}^U_{n,\alpha}(\omega)]$

Let $\tilde{R}(t) = \sup_{\alpha \in (0,1] \cap Q} \left\{\alpha \cdot I_{[\sum_{n=1}^{\tilde{N}^L_\alpha(t)} \tilde{R}^L_{n,\alpha}, \sum_{n=1}^{\tilde{N}^U_\alpha(t)} \tilde{R}^U_{n,\alpha}]}\right\}$, with Q being the set of the rational numbers,

then $\tilde{R}(t)$ represents the total reward earned by time t with inter-arrival times and a reward in the fuzzy sense.

For each $\alpha \in (0,1]$, $\tilde{R}(t)^L_\alpha$ and $\tilde{R}(t)^U_\alpha$ are defined by as follows:

$$\tilde{R}(t)^L_\alpha = \inf\{s \mid \tilde{R}(t)(s) \geq \alpha$$

and $$\tilde{R}(t)^U_\alpha = \sup\{s \mid \tilde{R}(t)(s) \geq \alpha\}$$

Hence, the α-level-set $[\tilde{R}(t)(\omega)]_\alpha = [\tilde{R}(t)^L_\alpha(\omega), \tilde{R}(t)^U_\alpha(\omega)]$.

The following Lemma is well-known in the Classical Analysis.

Lemma 4.1

Let $\{f_n\}$ be a sequence of monotonic functions on [0,1]. If $\{f_n\}$ converges pointwise to a continuous function f(x) on [0,1], then $f_n(x)$ converges to f(x) uniformly.

Let $F_c(R) = \{\tilde{u} \in F(R) \mid \tilde{u}^L_\alpha$ and $\tilde{u}^U_\alpha$ are continuous when considered as function of α

Proposition 4.1.

If $E\widetilde{R}_1$ and $E\widetilde{X}_1 \in F_c(R)$, then $d(\frac{\widetilde{R}(t)}{t}, \frac{E\widetilde{R}_1}{E\widetilde{X}_1}) \to 0$ as $t \to \infty$ a.s.

Proof: For any $0<\alpha \le 1$, we get the α -level-set $[\widetilde{R}(t)]_\alpha = [\widetilde{R}(t)_\alpha^L, \widetilde{R}(t)_\alpha^U]$

It is clear that we get $\widetilde{R}(t)_\alpha^L = \sum_{n=1}^{\widetilde{N}_\alpha^L(t)} \widetilde{R}_{n,\alpha}^L$ and $\widetilde{R}(t)_\alpha^U = \sum_{n=1}^{\widetilde{N}_\alpha^U(t)} \widetilde{R}_{n,\alpha}^U$

By the strong law of large numbers , we obtain that

$$\lim_{\widetilde{N}_\alpha^L(t)\to\infty} \frac{\sum_{n=1}^{\widetilde{N}_\alpha^L(t)} \widetilde{R}_{n,\alpha}^L}{\widetilde{N}_\alpha^L(t)} = E\widetilde{R}_{1,\alpha}^L \quad \text{a.s.}$$

Similarly ,we have
$$\lim_{\widetilde{N}_\alpha^L(t)\to\infty} \frac{\widetilde{N}_\alpha^L(t)}{\widetilde{S}_{\widetilde{N}_\alpha^L(t)}(\omega)_\alpha^U} = \frac{1}{E\widetilde{X}_{1,\alpha}^U} \quad \text{a.s.}$$

$$\lim_{\widetilde{N}_\alpha^L(t)\to\infty} \frac{\widetilde{N}_\alpha^L(t)}{\widetilde{S}_{\widetilde{N}_\alpha^L(t)+1}(\omega)_\alpha^U} = \frac{1}{E\widetilde{X}_{1,\alpha}^U} \quad \text{a.s.}$$

Let $\{r_k$ be a countable dense subset of $[0,1]$.Then there exist $B_k \in \Gamma$ with $P(B_k)=0$ such that for each $\omega \notin B_k$

$$\lim_{\widetilde{N}_{r_k}^L(t)\to\infty} \frac{\sum_{n=1}^{\widetilde{N}_{r_k}^L(t)} \widetilde{R}_{n,r_k}^L}{\widetilde{N}_{r_k}^L(t)} = E\widetilde{R}_{1,r_k}^L, \tag{4.1}$$

$$\lim_{\widetilde{N}_{r_k}^L(t)\to\infty} \frac{\widetilde{N}_{r_k}^L(t)}{\widetilde{S}_{\widetilde{N}_{r_k}^L(t)}(\omega)_{r_k}^U} = \frac{1}{E\widetilde{X}_{1,r_k}^U} \tag{4.2}$$

and
$$\lim_{\widetilde{N}_{r_k}^L(t)\to\infty} \frac{\widetilde{N}_{r_k}^L(t)}{\widetilde{S}_{\widetilde{N}_{r_k}^L(t)}(\omega)_{r_k}^U} = \frac{1}{E\widetilde{X}_{1,r_k}^U} \tag{4.3}$$

If we define $B=\bigcup_{k=0}^{\infty} B_k$, then $P(B)=0$ and for each $\omega \notin B$, (4.1),(4.2) and (4.3) hold for all r_k .New, we will show that for each $\omega \notin B$, $\lim_{\widetilde{N}_\alpha^L(t)\to\infty} \frac{\sum_{n=1}^{\widetilde{N}_\alpha^L(t)} \widetilde{R}_{n,\alpha}^L}{\widetilde{N}_\alpha^L(t)} = E\widetilde{R}_{1,\alpha}^L$ uniformly in $\alpha \in [0,1]$.Let $\omega \notin B$, $\beta \in (0,1)$ and $\varepsilon > 0$ be arbitrarily fixed.Then, by the continuity of $E\widetilde{R}_{1,\alpha}^L$ at β as a function of α, there exists $\delta > 0$ such that $|\alpha-\beta|<\delta$ implies

$$|E\widetilde{R}_{1,\alpha}^L - E\widetilde{R}_{1,\beta}^L| < \varepsilon .$$

If we take r_l, r_m so that $\beta-\delta < r_l < \beta < r_m < \beta+\delta$, then $E\widetilde{R}_{1,r_m}^L - \varepsilon < E\widetilde{R}_{1,\beta}^L < E\widetilde{R}_{1,r_l}^L + \varepsilon$.

Hence, by the monotonicity of $\frac{\sum_{n=1}^{\widetilde{N}_\alpha^L(t)} \widetilde{R}_{n,\alpha}^L}{\widetilde{N}_\alpha^L(t)}$ with respect to α ,

$$\frac{\sum_{n=1}^{\widetilde{N}_{r}^{L}(t)}\widetilde{R}_{n,r}^{L}}{\widetilde{N}_{r}^{L}(t)} - E\widetilde{R}_{1,r}^{L} - \varepsilon < \frac{\sum_{n=1}^{\widetilde{N}_{\beta}^{L}(t)}\widetilde{R}_{n,\beta}^{L}}{\widetilde{N}_{\beta}^{L}(t)} - E\widetilde{R}_{1,\beta}^{L} < \frac{\sum_{n=1}^{\widetilde{N}_{r_m}^{L}(t)}\widetilde{R}_{n,r_m}^{L}}{\widetilde{N}_{r_m}^{L}(t)} - E\widetilde{R}_{1,r_m}^{L} + \varepsilon$$

which implies $\dfrac{\sum_{n=1}^{\widetilde{N}_{\beta}^{L}(t)}\widetilde{R}_{n,\beta}^{L}}{\widetilde{N}_{\beta}^{L}(t)} \to E\widetilde{R}_{1,\beta}^{L}$.

Therefore, if $\omega \notin B$, then $\lim\limits_{\widetilde{N}_{\alpha}^{L}(t)\to\infty} \dfrac{\sum_{n=1}^{\widetilde{N}_{\alpha}^{L}(t)}\widetilde{R}_{n,\alpha}^{L}}{\widetilde{N}_{\alpha}^{L}(t)} = E\widetilde{R}_{1,\alpha}^{L}$ for each $\alpha \in [0,1]$.

Since $\dfrac{\sum_{n=1}^{\widetilde{N}_{\alpha}^{L}(t)}\widetilde{R}_{n,\alpha}^{L}}{\widetilde{N}_{\alpha}^{L}(t)}$ are monotonic functions on $[0,1]$ and $\dfrac{\sum_{n=1}^{\widetilde{N}_{\alpha}^{L}(t)}\widetilde{R}_{n,\alpha}^{L}}{\widetilde{N}_{\alpha}^{L}(t)}$ converges pointwise a continuous function $E\widetilde{R}_{1,\alpha}^{L}$ on $[0,1]$, it follows from Lemma 4.1 that the convergence is uniformly in $\alpha \in [0,1]$.

Similarly it can be proved that for each $\omega \notin B$,

$$\lim_{\widetilde{N}_{\alpha}^{L}(t)\to\infty} \frac{\widetilde{N}_{\alpha}^{L}(t)}{\widetilde{S}_{\widetilde{N}_{\alpha}^{L}(t)}(\omega)_{\alpha}^{U}} = \frac{1}{E\widetilde{X}_{1,\alpha}^{U}}$$

$$\text{and} \lim_{\widetilde{N}_{\alpha}^{L}(t)\to\infty} \frac{\widetilde{N}_{\alpha}^{L}(t)}{\widetilde{S}_{\widetilde{N}_{\alpha}^{L}(t)+1}(\omega)_{\alpha}^{U}} = \frac{1}{E\widetilde{X}_{1,\alpha}^{U}} \quad \text{uniformly } \alpha \in [0,1].$$

And for all $t > 0$, we have $\dfrac{\widetilde{N}_{\alpha}^{L}(t)}{\widetilde{S}_{\widetilde{N}_{\alpha}^{L}(t)+1}(\omega)_{\alpha}^{U}} \le \dfrac{\widetilde{N}_{\alpha}^{L}(t)}{t} \le \dfrac{\widetilde{N}_{\alpha}^{L}(t)}{\widetilde{S}_{\widetilde{N}_{\alpha}^{L}(t)}(\omega)_{\alpha}^{U}}$.

Then, we have that for each $\omega \notin B$, $\lim\limits_{t\to\infty} \dfrac{\widetilde{N}_{\alpha}^{L}(t)}{t} = \dfrac{1}{E\widetilde{X}_{1,\alpha}^{U}}$ uniformly $\alpha \in [0,1]$.

Hence, we obtain that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\widetilde{R}(t)_{\alpha}^{L}}{t} &= \lim_{t\to\infty} \frac{\widetilde{R}(t)_{\alpha}^{L}}{\widetilde{N}_{\alpha}^{L}(t)} \frac{\widetilde{N}_{\alpha}^{L}(t)}{t} \\
&= \lim_{t\to\infty} \frac{\sum_{n=1}^{\widetilde{N}_{\alpha}^{L}(t)}\widetilde{R}_{n,\alpha}^{L}}{\widetilde{N}_{\alpha}^{L}(t)} \frac{\widetilde{N}_{\alpha}^{L}(t)}{t} = \lim_{\widetilde{N}_{\alpha}^{L}(t)\to\infty} \frac{\sum_{n=1}^{\widetilde{N}_{\alpha}^{L}(t)}\widetilde{R}_{n,\alpha}^{L}}{\widetilde{N}_{\alpha}^{L}(t)} \lim_{t\to\infty} \frac{\widetilde{N}_{\alpha}^{L}(t)}{t} \\
&= \frac{E\widetilde{R}_{1,\alpha}^{L}}{E\widetilde{X}_{1,\alpha}^{U}} \quad \text{uniformly } \alpha \in [0,1].
\end{aligned}$$

Similarly, we have $\lim\limits_{t\to\infty} \dfrac{\widetilde{R}(t)_{\alpha}^{U}}{t} = \dfrac{E\widetilde{R}_{1,\alpha}^{U}}{E\widetilde{X}_{1,\alpha}^{L}}$ uniformly $\alpha \in [0,1]$.

Therefore, for each $\omega \notin B$, $d\left(\dfrac{\widetilde{R}(t)}{t}, \dfrac{E\widetilde{R}_{1}}{E\widetilde{X}_{1}}\right) \to 0$ as $t \to \infty$. a.s.

Proposition 4.1 states that the long run average fuzzy reward with inter-arrival time in the fuzzy sense is just the expected reward earned during a cycle divided by the expected length of the cycle in the fuzzy sense.

Example 4.1

John has a radio that works on a single battery. As soon as the battery in use fails, he immediately replaces it with a new battery. Let (Ω, Γ, P) be a probability space. We consider the lifetimes of batteries as a sequence of independent and identical distributed fuzzy random variables $\{\widetilde{X}_n, n \geq 1\}$ such that $\widetilde{X}_n : \Omega \to F(R^+)$, where $F(R^+)$ is the set of fuzzy numbers in R^+ (i.e. for $\omega \in \Omega$, $\widetilde{X}_n(\omega)$ $F(R^+)$) and the membership function $\widetilde{X}_n(\omega)(t) = \max(1-(\frac{t-S}{K})^2, 0)$, t R where $(S,K) : \Omega \to R \times R$ is a random vector such that the expectation ES and the expectation EK exist $(S \geq K > 0)$, $n \geq 1$.

For each $\alpha \in (0,1]$, both $\widetilde{X}^L_{1,\alpha}$ and $\widetilde{X}^U_{1,\alpha}$, gotten by

$$\widetilde{X}_1(\omega)^L_\alpha = \inf\{t|\widetilde{X}_1(\omega)(t) \geq \alpha\} = S - K(1-\alpha)^{1/2}$$

and $\widetilde{X}_1(\omega)^U_\alpha = \sup\{t|\widetilde{X}_1(\omega)(t) \geq \alpha\} = S + K(1-\alpha)^{1/2}$, are sequences of independent and identical distributed random variables.

The expectation of the lifetime of a battery is the fuzzy number $E\widetilde{X}_1$ and

$$E\widetilde{X}_1(t) = \sup_{\alpha \in (0,1] \cap Q}\{\alpha \cdot I_{[EX^L_{1,\alpha}, EX^U_{1,\alpha}]}(t)\} = \sup_{\alpha \in (0,1] \cap Q}\{\alpha \cdot I_{[ES-EK(1-\alpha)^{1/2}, ES+EK(1-\alpha)^{1/2}]}(t)$$

$$= \max(1-(\frac{t-ES}{EK})^2, 0), \ t \quad R. \tag{4.1.1}$$

Hence, the α -level-set $EX_{1,\alpha} = [ES - EK(1-\alpha)^{1/2}, ES+EK(1-\alpha)^{1/2}]$. (4.1.2)

We also consider the prices of batteries as a sequence of independent and identical distributed fuzzy random variables $\{\widetilde{R}_n, n \geq 1\}$ such that $\widetilde{R}_n : \Omega \to F(R^+)$, where $F(R^+)$ is the set of fuzzy numbers in R (i.e. for ω Ω, $\widetilde{R}_n(\omega) \in F(R^+)$) and the membership function $\widetilde{R}_n(\omega)(t) = \max(1-(\frac{t-R}{L})^2, 0)$, $t \in R$ where $(R, L) : \Omega \to R \times R$ is a random vector such that the expectation ER and the expectation EL exist $(R \geq L > 0)$, $n \geq 1$.

For each $\alpha \in (0,1]$, both $\widetilde{R}^L_{1,\alpha}$ and $\widetilde{R}^U_{1,\alpha}$, gotten by

$$\widetilde{R}^L_{1,\alpha}(\omega) = \inf\{t \mid \widetilde{R}_1(\omega)(t) \geq \alpha = R - L(1-\alpha)^{1/2}$$

and $\widetilde{R}^U_{1,\alpha}(\omega) = \sup\{t \mid \widetilde{R}_1(\omega)(t) \geq \alpha = R + L(1-\alpha)^{1/2}$, are sequences of independent and identical distributed random variables. Let $\widetilde{N}(t)$ denote the fuzzy number of batteries that have failed by time t, and let $\widetilde{R}(t) = \sup_{\alpha \in (0,1] \cap Q}\left\{\alpha \cdot I_{[\sum_{n=1}^{\widetilde{N}^L_\alpha(t)} \widetilde{R}^L_{n,\alpha}, \sum_{n=1}^{\widetilde{N}^U_\alpha(t)} \widetilde{R}^U_{n,\alpha}]}\right\}$, Q is the set of the rational numbers, then $\widetilde{R}(t)$ represents the total costs in the fuzzy sense by time t. By Proposition 4.1, we have that the long run average fuzzy costs with inter-arrival time in the fuzzy sense is $\lim_{t \to \infty} \frac{\widetilde{R}(t)}{t}$ and

$$\lim_{t\to\infty} \frac{\tilde{R}(t)}{t} = \frac{E\tilde{R}_1}{E\tilde{X}_1} = \sup_{\alpha\in(0,1]\cap Q} \left\{ \alpha \cdot 1_{\left[\frac{E\tilde{R}_1}{E\tilde{X}_1}\right]_\alpha} \right\}$$

$$= \sup_{\alpha\in(0,1]\cap Q} \left\{ \alpha \cdot 1_{\left[\frac{ER-EL(1-\alpha)^{1/2}}{ES+EK(1-\alpha)^{1/2}}, \frac{ER+EL(1-\alpha)^{1/2}}{ES-EK(1-\alpha)^{1/2}}\right]} \right\}$$

where Q is the set of the rational number.

That is, the long run average fuzzy reward with inter-arrival time in the fuzzy sense is $\frac{E\tilde{R}_1}{E\tilde{X}_1}$ and the α -level-set

$$\frac{E\tilde{R}_{1,\alpha}}{E\tilde{X}_{1,\alpha}} = \left[\frac{ER-EL(1-\alpha)^{\frac{1}{2}}}{ES+EK(1-\alpha)^{\frac{1}{2}}}, \frac{ER+EL(1-\alpha)^{\frac{1}{2}}}{ES-EK(1-\alpha)^{\frac{1}{2}}}\right] \qquad (4.1.3)$$

By (4.1.2.) and (4.1.3.), corresponding to the α -level-set $[ES-EK(1-\alpha)^{1/2}, ES+EK(1-\alpha)^{1/2}]$ of the expectation of the lifetime of a battery period, the α -level-set of the long run average fuzzy reward with inter-arrival time in the fuzzy sense is $\left[\frac{ER-EL(1-\alpha)^{\frac{1}{2}}}{ES+EK(1-\alpha)^{\frac{1}{2}}}, \frac{ER+EL(1-\alpha)^{\frac{1}{2}}}{ES-EK(1-\alpha)^{\frac{1}{2}}}\right]$.

From (4.1.3), letting $\frac{ER-EL(1-\alpha)^{\frac{1}{2}}}{ES+EK(1-\alpha)^{\frac{1}{2}}} = y_1$, we obtain $\alpha = 1-\left[\frac{ER-y_1ES}{EL+y_1EK}\right]^2$.

For $0 \le \alpha \le 1$, we obtain $\frac{ER-EL}{ES+EK} \le y_1 \le \frac{ER}{ES}$.

Hence, we obtain

$$\frac{E\tilde{R}_1}{E\tilde{X}}(y) = 1-\left[\frac{ER-yES}{EL+yEK}\right]^2, \qquad \text{if } \frac{ER-EL}{ES+EK} \le y \le \frac{ER}{ES}.$$

Similarly, we obtain

$$\frac{E\tilde{R}_1}{E\tilde{X}}(y) = 1-\left[\frac{yES-ER}{EL+yEK}\right]^2, \qquad \text{if } \frac{ER}{ES} \le y \le \frac{ER+EL}{ES-EK}.$$

Hence, we obtain the membership function $\frac{E\tilde{R}_1}{E\tilde{X}_1}(y)$ of $\frac{E\tilde{R}_1}{E\tilde{X}_1}$ as follows:

$$\frac{E\tilde{R}_1}{E\tilde{X}_1}(y) = 1-\left[\frac{ER-yES}{EL+yEK}\right]^2, \qquad \text{if } \frac{ER-EL}{ES+EK} \le y \le \frac{ER+EL}{ES-EK}.$$

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